WEAK DISTRIBUTIVITY IMPLYING DISTRIBUTIVITY

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Abstract. Let $\mathbb{B}$ be a complete Boolean algebra. We show that if $\lambda$ is an infinite cardinal and $\mathbb{B}$ is weakly $(\lambda^\omega, \omega)$-distributive, then $\mathbb{B}$ is $(\lambda, 2)$-distributive. Using a similar argument, we show that if $\kappa$ is a weakly compact cardinal such that $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive.

1. Introduction

Given sets $A$ and $B$, $^A B$ denotes the set of functions from $A$ to $B$. In this article, $\lambda$ and $\kappa$ will denote ordinals, although usually they can be assumed to be infinite cardinals. As defined in [6], given $\lambda$ and $\kappa$, we say that a complete Boolean algebra $\mathbb{B}$ is $(\lambda, \kappa)$-distributive iff

$$\prod_{\alpha < \lambda} \sum_{\beta < \kappa} u_{\alpha, \beta} = \sum_{f : \lambda \rightarrow \kappa} \prod_{\alpha < \lambda} u_{\alpha, f(\alpha)}$$

for any $\langle u_{\alpha, \beta} \in \mathbb{B} : \alpha < \lambda, \beta < \kappa \rangle$. Given maximal antichains $A_1, A_2 \subseteq \mathbb{B}$, we say that $A_2$ refines $A_1$ iff $(\forall a_2 \in A_2)(\exists a_1 \in A_1) a_2 \leq_B a_1$. It is a fact that $\mathbb{B}$ is $(\lambda, \kappa)$-distributive iff each size $\lambda$ collection of maximal antichains in $\mathbb{B}$ each of size $\kappa$ has a common refinement. There is also a useful characterization in terms of forcing (which can be found in [6] as Theorem 15.38):

**Fact 1.1.** A complete Boolean algebra $\mathbb{B}$ is $(\lambda, \kappa)$-distributive iff

$$1 \Vdash_B (\forall f : \lambda \rightarrow \kappa) f \in \check{V}.$$

Unfortunately, the definition of weakly distributive varies in the literature (for example [7]). We will use the one given by Jech (see [6]). That is, we say that a complete Boolean algebra $\mathbb{B}$ is weakly $(\lambda, \kappa)$-distributive iff

$$\prod_{\alpha < \lambda} \sum_{\beta < \kappa} u_{\alpha, \beta} = \sum_{g : \lambda \rightarrow \kappa} \prod_{\alpha < \lambda} \sum_{\beta < g(\alpha)} u_{\alpha, \beta}.$$

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This definition has a natural characterization in terms of forcing. Given a set $X$ and $f, g: X \to \kappa$, we write $f \leq g$ iff $g$ everywhere dominates $f$. That is,

$$(\forall x \in X) \ f(x) \leq g(x).$$

**Fact 1.2.** A complete Boolean algebra $\mathbb{B}$ is weakly $(\lambda, \kappa)$-distributive iff

$$1 \Vdash_{\mathbb{B}} (\forall f : \lambda \to \kappa)(\exists g : \lambda \to \kappa) \ g \in \check{V} \land f \leq g.$$ 

We will show the following:

**Theorem (A).** Let $\lambda$ be an infinite cardinal. If

1) $\mathbb{B}$ is weakly $(\lambda^\omega, \omega)$-distributive,

then $\mathbb{B}$ is $(\lambda, 2)$-distributive.

**Theorem (B).** Let $\kappa$ be a weakly compact cardinal. If

1) $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and
2) $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$,

then $\mathbb{B}$ is $(\kappa, 2)$-distributive.

We will then discuss why Theorem (B) does not hold when we have $\kappa = \omega_1$ instead of $\kappa$ being weakly compact, and we will show one way to fix the situation using the tower number. Finally, we use the same idea using the tower number to prove a variation of Theorem (A) involving weak $(\lambda^\omega, \kappa)$-distributivity for $\kappa > \omega$.

2. Functions from $\lambda^\omega$ to $\omega$

The proof of the following lemma uses the fact that well-foundedness of trees is absolute. It is crucial, for what follows, that this lemma does not require $\omega\lambda \subseteq M$. See [4] for motivation and discussion.

**Lemma 2.1.** For each $A \subseteq \lambda$, there is a function $f : \omega\lambda \to \omega$ such that whenever $M$ is a transitive model of ZF such that $\lambda \in M$ and some $g : (\omega\lambda)^M \to \omega$ in $M$ satisfies

$$(\forall x \in (\omega\lambda)^M) \ f(x) \leq g(x),$$

then $A \in M$.

**Proof.** Fix $A \subseteq \lambda$. Define $f : \omega\lambda \to \omega$ by

$$f(x) := \begin{cases} 
 0 & \text{if } (\forall n < \omega) \ x(n) \not\in A, \\
 n + 1 & \text{if } x(n) \in A \text{ but } (\forall m < n) \ x(m) \not\in A.
\end{cases}$$
Let $M$ be a transitive model of ZF such that $\lambda \in M$ but $A \not\subseteq M$. Suppose, towards a contradiction, that there is some $g \in M$ satisfying $(\forall x \in (\omega \lambda)^M) f(x) \leq g(x)$. Let $B$ be the set

$$B := \{ t \in <\omega \lambda : g(x) \geq |t| \text{ for all } x \in M \text{ extending } t \}.$$ 

Notice that $B \in M$. Let $T \subseteq <\omega \lambda$ be the set of elements of $B$ all of whose initial segments are also in $B$. Note that $T$ is a tree and $T \in M$.

For all $a \in \lambda$, $a \in A$ implies $\langle a \rangle \in B$. Thus, there must be some $a_0 \in \lambda$ such that $a_0 \not\in A$ but $\langle a_0 \rangle \in B$. If there was not, then $A$ could be defined in $M$ by $A = \{ a \in \lambda : \langle a \rangle \in B \}$.

Next, for all $a \in \lambda$, $a \in A$ implies $\langle a_0, a \rangle \in B$. Thus, by similar reasoning as before, there must be some $a_1 \in \lambda$ such that $a_1 \not\in A$ but $\langle a_0, a_1 \rangle \in B$. Continuing like this, we can construct a sequence $x \in \omega \lambda$ satisfying $(\forall n < \omega) x \upharpoonright n \in B$. Thus, $(\forall n < \omega) g(x') \geq n$, which is impossible. □

This implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(\lambda^\omega)^M$ instead of the set of sequences $(\omega \lambda)^M$ as their domains:

**Lemma 2.2.** Let $M$ be a transitive model of ZF such that the ordinal $\lambda$ is in $M$ and $(\omega \lambda)^M$ can be well-ordered in $M$. Assume that for each $f : (\lambda^\omega)^M \to \omega$ there is some $g : (\lambda^\omega)^M \to \omega$ in $M$ such that $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.

**Proof.** Consider any $A \in \mathcal{P}(\lambda)$. Use the lemma above with $A$ to get $\tilde{f} : \omega \lambda \to \omega$ such that if $\tilde{g} : (\omega \lambda)^M \to \omega$ is any function in $M$ which satisfies

$$(1) \quad (\forall x \in (\omega \lambda)^M) \tilde{f}(x) \leq \tilde{g}(x),$$

then $A \in M$. Since $(\omega \lambda)^M$ can be well-ordered in $M$, fix a bijection

$$\eta : (\lambda^\omega)^M \to (\omega \lambda)^M$$

in $M$. Define $f : (\lambda^\omega)^M \to \omega$ by

$$f(\alpha) := \tilde{f}(\eta(\alpha)).$$
That is, the following diagram commutes:

\[
\begin{array}{ccc}
(\omega \lambda)^M \xrightarrow{f} \omega \\
\eta \uparrow \supseteq \downarrow \swarrow f \\
(\lambda \omega)^M
\end{array}
\]

By hypothesis, let \( g : (\lambda \omega)^M \to \omega \) be a function in \( M \) which everywhere dominates \( f \). Define \( \tilde{g} : (\omega \lambda)^M \to \omega \) by

\[
\tilde{g}(x) := g(\eta^{-1}(x)).
\]

We have that \( \tilde{g} \in M \) and \( \tilde{g} \) satisfies [1] so by the hypothesis on \( \tilde{f} \), \( A \in M \).

We now have the main result of this section:

**Theorem (A).** Let \( \mathbb{B} \) be a complete Boolean algebra and \( \lambda \) be an infinite cardinal. If \( \mathbb{B} \) is weakly \((\lambda^\omega, \omega)\)-distributive, then \( \mathbb{B} \) is \((\lambda, 2)\)-distributive.

**Proof.** Let \( \mu := \lambda^\omega \). Assume \( \mathbb{B} \) is weakly \((\mu, \omega)\)-distributive. Force with \( \mathbb{B} \). Every \( f : \mu \to \omega \) in the extension can be everywhere dominated by some \( g : \mu \to \omega \) in the ground model, so applying the lemma above in the extension (setting \( M \) to be the ground model) tells us that the \( \mathcal{P}(\lambda) \) of the extension is included in the ground model. Hence, \( \mathbb{B} \) is \((\lambda, 2)\)-distributive.

### 3. Functions from \( 2^\kappa \) to \( \kappa \) with \( \kappa \) Weakly Compact

The first lemma in the previous section was the key to the theorem there. We have a parallel lemma here which, instead of using the absoluteness of trees being well-founded, uses the tree property to get similar absoluteness. It is important that this lemma does not require \( \kappa^2 \subseteq M \). By weakly compact, we mean strongly inaccessible and having the tree property.

**Lemma 3.1.** For each \( a \in \kappa^2 \), there is a function \( f : \kappa^2 \to \kappa \) such that whenever \( M \) is a transitive model of ZF such that \( \kappa \in M \), \( \kappa^2 \subseteq M \), \((\kappa^2)^M\) can be well-ordered in \( M \), \((\kappa \text{ is weakly compact})^M\), and some \( g : (\kappa^2)^M \to \kappa \) in \( M \) satisfies

\[
(\forall x \in (\kappa^2)^M) \ f(x) \leq g(x),
\]

then \( a \in M \).
Proof. Fix $a \in {}^\kappa 2$. Let $f : {}^\kappa 2 \to \kappa$ be the function

$$f(x) := \begin{cases} 
0 & \text{if } (\forall \alpha < \kappa) x(\alpha) = a(\alpha), \\
\alpha + 1 & \text{if } x(\alpha) \neq a(\alpha) \text{ but } (\forall \beta < \alpha) x(\beta) = a(\beta).
\end{cases}$$

Let $M$ be an appropriate transitive model of ZF. Suppose $g : (\kappa^2)^M \to \kappa$ in $M$ satisfies $(\forall x \in (\kappa^2)^M) f(x) \leq g(x)$. We will show that $a \in M$.

Suppose, towards a contradiction, that $a \notin M$. Let

$$B := \{t \in {}^{<\kappa}2 : g(x) \geq \text{Dom}(t) \text{ for all } x \in M \text{ extending } t\}.$$ 

Note that by definition, there cannot be any $x \in {}^\kappa 2$ in $M$ satisfying $(\forall \alpha < \kappa) x \upharpoonright \alpha \in B$ because if there was such an $x$, we would have $(\forall \alpha < \kappa) g(x(\alpha)) \geq \alpha$, which is impossible. Since $B$ need not be a tree, let $T \subseteq {}^{<\kappa}2$ be the tree of those elements of $B$ all of whose initial segments are also in $B$. Again, $T$ cannot have a length $\kappa$ path in $M$. Note that for each $\alpha < \kappa$, $a \upharpoonright \alpha \in B$. This is because any $x \in {}^\kappa 2$ in $M$ which extends $a \upharpoonright \alpha$ differs from $a$ (since $a \notin M$), and the smallest $\gamma$ such that $x(\gamma) \neq a(\gamma)$ must be $\geq \alpha$, so $g(x) \geq f(x) = \gamma + 1 > \gamma \geq \alpha = \text{Dom}(a \upharpoonright \alpha)$.

Since $(\forall \alpha < \kappa) a \upharpoonright \alpha \in B$, also $(\forall \alpha < \kappa) a \upharpoonright \alpha \in T$.

Now, $B \in M$ (since $<\kappa 2 \subseteq M$ and $g \in M$) and so $T \in M$. Since $(\forall \alpha < \kappa) a \upharpoonright \alpha \in T$, $T$ has height $\kappa^M$. Since $(\kappa$ is strongly inaccessible)$^M$, we have $(T$ is a $\kappa$-tree)$^M$. Since $(\kappa$ has the tree property)$^M$, there is a length $\kappa$ path through $T$ in $M$, which we said earlier was impossible.\]

As before, this implies the following lemma, whose order of quantifiers is not as powerful, but the functions have the ordinal $(2^\kappa)^M$ instead of the set of sequences $(\kappa^2)^M$ as their domains:

**Lemma 3.2.** Let $M$ be a transitive model of ZF such that the ordinal $\kappa$ is in $M$, $<\kappa 2 \subseteq M$, $(\kappa^2)^M$ can be well-ordered in $M$, and $(\kappa$ is weakly compact)$^M$. Assume that for each $f : (2^\kappa)^M \to \kappa$ there is some $g : (2^\kappa)^M \to \kappa$ in $M$ such that $f \leq g$. Then $\mathcal{P}(\kappa) \subseteq M$.

**Proof.** The proof is similar to that of Lemma 2.2.\]

As before, the main result of this section follows:

**Theorem (B).** Let $\mathbb{B}$ be a complete Boolean algebra and $\kappa$ be a weakly compact cardinal. If $\mathbb{B}$ is weakly $(2^\kappa, \kappa)$-distributive and $\mathbb{B}$ is $(\alpha, 2)$-distributive for each $\alpha < \kappa$, then $\mathbb{B}$ is $(\kappa, 2)$-distributive.

**Proof.** This follows from the lemma above just as Theorem (A) followed from Lemma 2.2.\]
4. The Tower Number

One might hope that Theorem (B) holds when \( \kappa = \omega_1 \) instead of \( \kappa \) being weakly compact. That is, one might hope that if a complete Boolean algebra \( B \) is weakly \((2^{\omega_1}, \omega_1)\)-distributive and \((\omega, 2)\)-distributive, then it is \((\omega_1, 2)\)-distributive. Unfortunately, this cannot be proved in ZFC because \( B \) could be a Suslin algebra (a Suslin algebra is c.c.c. and therefore is weakly \((\lambda, \omega_1)\)-distributive for any \( \lambda \)). However, if we add the assumption that \( 1 \vDash B(\omega_1 < t) \), where we will define \( t \) soon, then \( B \) is \((\omega_1, 2)\)-distributive. The argument is simpler than that of Theorem (A) and Theorem (B) and does not need the hypothesis of weak \((2^{\omega_1}, \omega_1)\)-distributivity. As a final twist, we will combine several ideas to prove a variation of Theorem (A).

Recall that \( t \), the tower number, is the smallest length of a sequence

\[
\langle S_\alpha \in [\omega]^{\omega} : \alpha < \kappa \rangle
\]

satisfying \((\forall \alpha < \beta < \kappa) S_\alpha \supseteq^* S_\beta \) but there is no \( S \in [\omega]^{\omega} \) satisfying \((\forall \alpha < \kappa) S_\alpha \supseteq^* S \) (where \( S_1 \subseteq^* S_2 \) means \( S_1 - S_2 \) is finite). It is not hard to see that \( \omega_1 \leq t \leq 2^{\omega} \). See [1] for more on \( t \) and related cardinals. The following lemma is the key. The idea is borrowed from Farah in [3], who got the idea from Dordal in [2], who got the idea from Booth.

**Lemma 4.1.** Let \( \kappa \) be such that \( \omega_1 \leq \kappa < t \). Let \( M \) be a transitive model of ZFC such that \( \kappa \in M \) and \((\forall \alpha < \kappa) \mathcal{P}(\alpha) \subseteq M \). Then \( \mathcal{P}(\kappa) \subseteq M \).

**Proof.** Fix \( \kappa \) and \( M \). Since \( \kappa \in M \) and \((\forall \alpha < \kappa) \mathcal{P}(\alpha) \subseteq M \), we have \( <^{\kappa} 2 \subseteq M \). Let \( F : <^{\kappa} 2 \to [\omega]^{\omega} \) be a function in \( M \) such that for all \( t_1, t_2 \in <^{\kappa} 2 \),

1. \( t_1 \subseteq t_2 \Rightarrow F(t_1) \supseteq^* F(t_2) \), and
2. \( t_1 \perp t_2 \Rightarrow F(t_1) \cap F(t_2) \) is finite.

Such functions are easy to construct by induction (and the Axiom of Choice). The construction will not get stuck at a limit stage \( \gamma < \kappa \) because given \( t \in \gamma 2 \subseteq M \) and \( \langle F(t \upharpoonright \alpha) : \alpha < \gamma \rangle \), since \( \gamma < t \) there is some \( S \in [\omega]^{\omega} \subseteq M \) such that \((\forall \alpha < \gamma) S \subseteq^* F(t \upharpoonright \alpha) \). The set \( F(t) \) can be defined to be the least such \( S \) accding to some fixed well-ordering of \([\omega]^{\omega}\).

Now, consider any \( a \in <^{\kappa} 2 \). We will show that \( a \in M \). The sequence \( \langle F(a \upharpoonright \alpha) : \alpha < \kappa \rangle \) is a \( \supseteq^* \)-chain (in \( V \)) of length \( \kappa \). Since \( \kappa < t \), fix some \( S \in [\omega]^{\omega} \) satisfying

\[ (\forall \alpha < \kappa) S \subseteq^* F(a \upharpoonright \alpha). \]
Since $\mathcal{P}(\omega) \subseteq M$, in particular $S \in M$. Within $M$, the function $F$ and the set $S$ can be used together to define $a$:

$$a = \bigcup\{t \in <\kappa^2 : S \subseteq^* F(t)\}.$$  

□

By applying the lemma above inductively, we get an improvement:

**Lemma 4.2.** Let $\kappa$ be such that $\omega_1 \leq \kappa < t$. Let $M$ be a transitive model of ZFC such that $\mathcal{P}(\omega) \subseteq M$. Then $\mathcal{P}(\kappa) \subseteq M$.

This last lemma is closely related to the fact that $2^\kappa = 2^\omega$ whenever $\kappa < t$. A proof of this using an argument similar to Lemma 4.1 can be found in [1]. Martin’s Axiom (MA) implies $t = 2^\omega$, but the original proof [8] that MA implies $2^\kappa = 2^\omega$ whenever $\kappa < 2^\omega$ used the almost disjoint coding poset. We now have the application to complete Boolean algebras:

**Proposition 4.3.** Let $\kappa$ be an infinite cardinal. Let $B$ be a complete Boolean algebra such that $B$ is $(\omega, 2)$-distributive and $1 \Vdash_B (\check{\kappa} < t)$. Then $B$ is $(\kappa, 2)$-distributive.

**Proof.** Apply Lemma 4.2 in the forcing extension with $M$ equal to the ground model. □

Let $\kappa$ be such that $\omega_1 \leq \kappa < t$. Any $A \subseteq [\omega]^{\omega}$ can be partitioned into $2^\omega$ infinite sets with pairwise finite intersection. Thus, fixing $\lambda \leq 2^\omega$, the function $F : <\kappa^2 \rightarrow [\omega]^{\omega}$ in Lemma 4.1 can be replaced by a function $F : <\kappa \lambda \rightarrow [\omega]^{\omega}$ satisfying the same conditions. Slightly modifying the proof of Lemma 4.1, we get that if $M$ is a transitive model of ZFC such that $\lambda \in M$ and $(\forall \alpha < \kappa)^M \alpha \lambda \subseteq M$, then $^\kappa \lambda \subseteq M$. Inductively applying this fact yields an improvement:

**Lemma 4.4.** Let $\kappa$ and $\lambda$ be such that $\omega_1 \leq \kappa < t$ and $\lambda \leq 2^\omega$. Let $M$ be a transitive model of ZFC such that $\lambda \in M$ and $^\omega \lambda \subseteq M$. Then $^\kappa \lambda \subseteq M$.

Now we may combine Lemma 4.4 with the argument in Lemma 2.1. The case $\kappa = \omega$ of this next lemma is already handled by Lemma 2.1

**Lemma 4.5.** Let $\kappa$ and $\lambda$ be such that $\omega \leq \kappa < t$ and $\lambda \leq 2^\omega$. For each $A \subseteq \lambda$, there is a function $f : ^\omega \lambda \rightarrow \kappa$ such that whenever $M$ is a transitive model of ZFC such that $^\omega \lambda \subseteq M$ (and therefore $^\kappa \lambda \subseteq M$) and some $g : ^\omega \lambda \rightarrow \kappa$ in $M$ satisfies $f \leq g$, then $A \in M$.

**Proof.** Fix $\kappa$, $\lambda$, and $A$. Define $f : ^\omega \lambda \rightarrow \kappa$ by

$$f(x) := \begin{cases} 0 & \text{if } (\forall \alpha < \kappa) x(\alpha) \notin A, \\ \alpha + 1 & \text{if } x(\alpha) \in A \text{ but } (\forall \beta < \alpha) x(\beta) \notin A. \end{cases}$$
This is the analogue of the function $f$ defined in Lemma 2.1. Now fix $M$ and some $g : {}^\kappa \lambda \to \kappa$ in $M$ satisfying $f \leq g$. Note that $2^\omega \in M$ so therefore $\kappa, \lambda \in M$. Let

$$B := \{ t \in {}^\kappa \lambda : g(x) \geq \text{Dom}(t) \text{ for all } x \text{ extending } t \}.$$ 

Since ${}^\kappa \lambda \cup \{ \kappa, \lambda, g \} \subseteq M$, also $B \in M$.

Assume towards a contradiction, that $A \not\in M$. Arguing just as in Lemma 2.1 there is some $x \in {}^\kappa \lambda$ satisfying $(\forall \alpha < \kappa) x \upharpoonright \alpha \in B$. Since $\kappa \lambda \subseteq M$, we have $x \in M$, and in particular $x$ is in the domain of $g$. We now have $(\forall \alpha < \kappa) g(x) \geq \alpha$, which is impossible. \hfill $\square$

**Lemma 4.6.** Let $\kappa$ and $\lambda$ be such that $\omega \leq \kappa < t$ and $\lambda \leq 2^\omega$. Let $M$ be a transitive model of ZFC such that $\omega \lambda \subseteq M$ (and therefore $\kappa \lambda \subseteq M$). Assume that for each $f : (\lambda^\kappa)^M \to \kappa$ there is some $g : (\lambda^\kappa)^M \to \kappa$ in $M$ satisfying $f \leq g$. Then $\mathcal{P}(\lambda) \subseteq M$.

**Proof.** This follows immediately from the previous lemma. \hfill $\square$

Now follows the theorem:

**Theorem 4.7.** Let $\mathbb{B}$ be a complete Boolean algebra. Let $\kappa$ and $\lambda$ be such that $1 \models_{\mathbb{B}} (\kappa < t)$ and $1 \models_{\mathbb{B}} (\lambda \leq 2^\omega)$. Assume that $\mathbb{B}$ is $(\omega, \lambda)$-distributive and weakly $(\lambda^\kappa, \kappa)$-distributive. Then $\mathbb{B}$ is $(\lambda, 2)$-distributive.

**References**


