ABSTRACT. This is a followup to a paper by the author where the disjointness relation for definable functions from \( \omega \) to \( \omega \) is analyzed. In that paper, for each \( a \in \omega \) we defined a Baire class one function \( f_a : \omega \to \omega \) which encoded \( a \) in a certain sense. Given \( g : \omega \to \omega \), let \( \Psi(g) \) be the statement that \( g \) is disjoint from at most countably many of the functions \( f_a \). We show the consistency strength of \((\forall g) \Psi(g)\) is that of an inaccessible cardinal. We show that AD\(^+\) implies \((\forall g) \Psi(g)\. Finally, we show that assuming large cardinals, \((\forall g) \Psi(g)\) holds in models of the form \( L(\mathbb{R})[\mathcal{U}] \) where \( \mathcal{U} \) is a selective ultrafilter on \( \omega \).

1. Introduction

In [2], for each \( a \in \omega \) we defined a Baire class one function \( f_a : \omega \to \omega \) with the intuition that a “nice” function can only be disjoint from \( f_a \) by “knowing about \( a \)”. We will review the definition of \( f_a \) in Section 3. We showed that assuming large cardinals, if \( g : \omega \to \omega \) is projective and \( g \cap f_a = \emptyset \), then \( a \) is in a countable set associated to \( g \). Hence, a projective \( g \) can be disjoint from at most countably many \( f_a \)'s. In what follows, PD stands for Projective Determinacy. We showed the following:

**Fact 1.1.** Assume PD. Fix a pointclass \( \Gamma \) and a function \( g : \omega \to \omega \) in \( \Gamma \). Let \( c \in \omega \) be a code for \( g \). Assume \( g \cap f_a = \emptyset \).

- \( \Gamma = \Delta_1^1 \Rightarrow a \in \Delta_1^1(c) \).
- \( \Gamma = \Delta_2^1 \Rightarrow a \in L[c] \).
- \( (\forall n \in \omega) \Gamma = \Delta_{3+n}^1 \Rightarrow a \in \mathcal{M}_{1+n}(c) \).

The \( \Gamma = \Delta_1^1 \) case is true in ZFC alone. We proved the \( \Gamma = \Delta_2^1 \) case assuming that \( \omega_1 \) is inaccessible in \( L[c] \). In Section 5 we will show that this assumption can be removed. In the \( \Gamma = \Delta_{3+n}^1 \) case for \( n \geq 3 \) we assume that \( \mathcal{M}_{n-2}(c) \) exists, that \( \omega_1 \) is inaccessible in this model, and that its forcing extensions by a certain small forcing \( H \) can compute

A portion of the results of this paper were proven during the September 2012 Fields Institute Workshop on Forcing while the author was supported by the Fields Institute. Work was also done while under NSF grant DMS-0943832.
\( \Sigma^1_n(c) \) truth. Here, \( \mathcal{M}_n(c) \) is a canonical inner model with \( n \) Woodin cardinals and containing \( c \). The requirement that \( \omega_1 \) be inaccessible is only needed to get the collection of dense subsets of \( \mathbb{H} \) in the inner model to be countable in \( V \).

Note that assuming PD, we have that \( a \) is \( \Delta^1_2 \) in \( c \) and a countable ordinal iff \( a \in L[c] \), and for \( n \geq 3 \), \( a \) is \( \Delta^1_n \) in \( c \) and a countable ordinal iff \( a \in \mathcal{M}_{n-2}(c) \). Thus, we may succinctly write the following:

**Fact 1.2.** Assume PD. Let \( 1 \leq n < \omega \). Let \( g \) be a \( \Delta^1_n(c) \) function for some \( c \in \omega^\omega \). Then \( g \cap f_a = \emptyset \) implies \( a \) is \( \Delta^1_n \) in \( c \) and a countable ordinal.

In this paper, we extend this result about projective functions to all functions \( g \) in models of \( \text{AD}^+ \) containing all the reals. Hence, assuming there is a proper class of Woodin cardinals, our results apply to all universally Baire functions. This suggests introducing a regularity property:

**Definition 1.3.** Given \( g : \omega^\omega \to \omega^\omega \), \( \Psi(g) \) is the statement that \( g \) is disjoint from at most countably many of the functions \( f_a \).

By Fact 1.1, PD implies \( \Psi(g) \) for every projective \( g : \omega^\omega \to \omega^\omega \). We will show that \( \text{AD}^+ \) implies \( \Psi(g) \) for all \( g : \omega^\omega \to \omega^\omega \). We also show, assuming large cardinals, that \( \Psi(g) \) holds for every \( g \) in models of the form \( L(\mathbb{R})[\mathcal{U}] \), where \( \mathcal{U} \) is a selective ultrafilter on \( \omega \). On the other hand, we will soon see that the existence of a well-ordering of \( \mathbb{R} \) implies \( \neg \Psi(g) \) for some \( g \).

The reader might wonder if the family \( \{ f_a : a \in \omega^\omega \} \) is really the right one to consider. We introduce the following statement \( \Psi \) to address the general situation. It follows that if \( (\forall g : \omega^\omega \to \omega^\omega) \Psi(g) \), then \( \Psi \).

**Definition 1.4.** \( \Psi \) is the statement that there is a family \( \{ f^a : a \in \omega^\omega \} \) of functions from \( \omega^\omega \) to \( \omega^\omega \) such that

1) The function \( (a, x) \mapsto f^a(x) \) is Borel;
2) No \( g : \omega^\omega \to \omega^\omega \) is disjoint from uncountably many of the \( f^a \).

We write \( a \) as a superscript in the \( f^a \) above to differentiate it from the specific function \( f_a \). Condition 1) is not the essential point, although without it we are left with a more combinatorial notion. We believe that in any natural setting where \( \Psi \) holds, then the family \( \{ f_a : a \in \omega^\omega \} \) witnesses this. In Section 2 we will show that the weakening of \( \Psi \) where 2) only applies to \( \Delta^1_2 \) functions \( g \) implies \( \omega_1 \) is inaccessible in \( L[r] \) for all \( r \in \omega^\omega \). Combining this with Fact 1.1, we get that the following are equivalent:

- \( (\forall g \in \Delta^1_2) \Psi(g) \).
In Section 6 we will show that \((\forall g) \Psi(g)\) holds in the Solovay model. Hence, the consistency strength of \((\forall g) \Psi(g)\) is that of an inaccessible cardinal.

Recall that Uniformization is the fragment of the Axiom of Choice that states that given any \(R \subseteq \omega \times \omega\) satisfying \((\forall x \in \omega)(\exists y \in \omega)(x, y) \in R\), then there is a function \(u : \omega \rightarrow \omega\) such that \(u \subseteq R\). We call \(u\) a uniformization for \(R\), or say that \(R\) is uniformized by \(u\).

As a convention, whenever we write \(ZF\) explicitly as a hypothesis to a lemma or a proposition etc, then we will not be assuming the Axiom of Choice. Otherwise, we will be.

**Proposition 1.5.**  \(ZF + \text{Uniformization} + \Psi\) implies that if \(S \subseteq \omega\) is uncountable, then it can be surjected onto \(\omega\) by a Borel function.

**Proof.** Fix an uncountable set \(S \subseteq \omega\). For each \(x \in \omega\), the function \(a \mapsto f^a(x)\) is Borel. We claim that for some \(x \in \omega\), the function \(a \mapsto f^a(x)\) surjects \(S\) onto \(\omega\). Suppose this is not the case. For each \(x \in \omega\), the set \(Y_x := \omega - \{f^a(x) : a \in S\}\) is non-empty. Apply Uniformization to get \(g : \omega \rightarrow \omega\) such that \((\forall x \in \omega)(g(x) \in Y_x)\). Then \(g\) is disjoint from \(f^a\) for each \(a \in S\), which is a contradiction because \(g\) can be disjoint from only countably many of the \(f^a\) functions. \(\square\)

Recall the statement PSP, the perfect set property, which states that every uncountable set of reals has a perfect subset. A set of reals is perfect iff it is non-empty and equal to its set of limit points. One can verify that if an uncountable set \(S \subseteq \omega\) has a perfect subset, then \(S\) can be surjected onto \(\omega\). This suggests that \(\Psi\) is related to PSP. Another indication of a connection is that our proof that \(L([\mathbb{R}])[\mathcal{U}]\) satisfies \(\Psi\) uses the fact that it satisfies PSP.

1.1. **\(\Psi\) is inconsistent with ZFC.** It is clear that \(\Psi\) is inconsistent with \(ZFC + \neg CH\), because given any \(S \subseteq \omega\) of size \(\omega_1 < 2^\omega\), there is a \(g\) disjoint from \(f_a\) for each \(a \in S\). Now assume \(ZFC + CH + \Psi\). We will prove a contradiction. By Proposition 1.5, every uncountable \(S \subseteq \omega\) can be surjected onto \(\omega\) by a Borel function. Hence, every \(S \subseteq \omega\) of size \(2^\omega\) can be surjected onto \(\omega\) by a Borel function. Recall that \(\text{add}(\mathcal{B})\) is the smallest size of a collection of meager sets of reals whose union is not meager. We have \(\omega_1 \leq \text{add}(\mathcal{B}) \leq 2^\omega\). This next proposition gives us our contradiction. Paul Larson pointed out how to make the diagonalization not get stuck by using the meager ideal.
Proposition 1.6. Assume ZFC + \( \text{add}(\mathcal{B}) = 2^{\omega} \), which is implied by CH. Then there exists a size \( 2^{\omega} \) set \( S \subseteq ^\omega \omega \) that cannot be surjected onto \( ^\omega \omega \) by any Borel function.

Proof. Because \( \text{add}(\mathcal{B}) = 2^{\omega} \), the union of \( < 2^{\omega} \) meager sets of reals is meager. For each Borel function \( h \) and each \( y \in ^\omega \omega \), \( h^{-1}(y) \) has the property of Baire, so it is either comeager below a basic open set or it is meager. There can be only countably many \( y \) such that \( h^{-1}(y) \) is meager. There can be only countably many \( y \) such that \( h^{-1}(y) \) is comeager below some basic open set, because otherwise there would be two that intersect.

We now begin the construction of \( S = \{a_\alpha : \alpha < 2^{\omega}\} \). Let \( \langle h_\alpha : \alpha < 2^{\omega}\rangle \) be an enumeration of all Borel functions from \( ^\omega \omega \) to \( ^\omega \omega \). First, pick any \( y_0 \in ^\omega \omega \) such that \( X_0 := h_0^{-1}(y_0) \) is meager. This \( y_0 \) will witness that \( h_0 \) does not surject \( S \) onto \( ^\omega \omega \). Now pick any \( a_0 \in ^\omega \omega - X_0 \).

At stage \( \alpha < 2^{\omega} \), pick any \( y_\alpha \in ^\omega \omega \) such that \( X_\alpha := h_\alpha^{-1}(y_\alpha) \) is meager and does not contain any \( a_\beta \) for \( \beta < \alpha \). This is possible because there are only \( < 2^{\omega} \) many \( y \) such that \( h_\alpha^{-1}(y) \) contains some \( a_\beta \) for \( \beta < \alpha \), and there are only \( \omega \) many \( y \) such that \( h_\alpha^{-1}(y) \) is not meager. Then pick \( a_\alpha \in ^\omega \omega - \{a_\beta : \beta < \alpha\} - \bigcup_{\beta < \alpha} X_\beta \). When the construction finishes, the set \( S \) will have size \( 2^{\omega} \) and for each \( \alpha < 2^{\omega} \), \( y_\alpha \not\in h_\alpha(A) \). \( \square \)

Corollary 1.7. ZFC implies \( \neg \Psi \).

Remark 1.8. Miller [6] has shown that in the iterated perfect set model, in which \( \omega_1 = \text{add}(\mathcal{B}) < \omega_2 = 2^{\omega} \), every size \( \omega_2 \) set \( S \subseteq ^\omega \omega \) can be surjected onto \( ^\omega \omega \) by a continuous function. The iterated perfect set model is obtained by starting with a model of CH and then adding \( \omega_2 \) many Sacks reals by a countable support iteration. This leads us to the following question:

Question 1.9. Suppose we weaken part 2) in Definition 1.4 to only require \( g \) to be disjoint from \( < 2^{\omega} \) of the \( f^a \). Is this weaker statement consistent with ZFC?

Remark 1.10. This is a different type of information that the disjointness relation can capture. Namely, assume AD. Fix \( \alpha < \Theta \), where \( \Theta \) is the smallest ordinal that \( ^\omega \omega \) cannot be surjected onto. Then there is a function \( f : ^\omega \omega \to ^\omega \omega \) such that if \( g : ^\omega \omega \to ^\omega \omega \) is any function that satisfies \( g \cap f = \emptyset \), then \( g \) has Wadge rank \( > \alpha \). We can construct \( f \) by diagonalizing over all functions of Wadge rank \( \leq \alpha \): let \( \langle h_x : x \in ^\omega \omega \rangle \) be an canonical enumeration of all continuous functions from \( ^\omega \omega \times ^\omega \omega \) to \( ^\omega \omega \). Let \( W \subseteq ^\omega \omega \) be a set of Wadge rank \( \alpha \). For each \( x \in ^\omega \omega \), if \( h_x^{-1}(W) \) is a function, define \( f(x) := h_x^{-1}(W)(x) \). Otherwise, define \( f(x) \) to be anything. Every Wadge rank \( \leq \alpha \) function from \( ^\omega \omega \) to \( ^\omega \omega \) appears as some \( h_x^{-1}(W) \).
Since $\omega \omega \cong \omega \omega \upharpoonright \omega \omega$, we may combine this remark with Theorem 4.4 that we will prove. That is, assume $\text{AD}^+$. For every $\alpha < \Theta$ and for every $a \in \omega \omega$, there is a function $f : \omega \omega \to \omega \omega$ such that whenever $g : \omega \omega \to \omega \omega$ satisfies $g \cap f = \emptyset$, then

1) $g$ has Wadge rank $> \alpha$, and
2) $a \in L[C]$ for any $\infty$-Borel code $C$ for $g$.

2. Consistency Strength Lower Bound

In the introduction we gave an argument that ZFC implies $\neg \Psi$. Using that argument and being careful about the complexity of the objects being produced yields a proof that $V = L$ implies $\neg \Psi(g)$ for some $\Delta^1_2$ function $g$. On the other hand, by Fact [1.1], ZFC proves $(\forall g \in \Delta^1_2) \Psi(g)$. Indeed, this is a theorem of ZF. Not only does $V = L$ imply $\neg \Psi(g)$ for some $\Delta^1_2$ function $g$, it also implies that for any family $\{f^n : a \in \omega \omega\}$ such that $(a, x) \mapsto f^n(x)$ is Borel, there is some $\Delta^1_2$ function disjoint from uncountably many of the $f^n$ functions. Relativizing this to $V = L[r]$ and using Shoenfield’s absoluteness theorem yields a proof that $ZF + DC + \Psi$ implies $\omega_1$ is inaccessible in $L[r]$ for each $r \in \omega \omega$.

Temporarily suppose $\Gamma$ is a pointclass closed under quantification of natural numbers. Let $\Delta = \Gamma \cap \neg \Gamma$. Let $g : \omega \omega \to \omega \omega$. Consider the ternary relation “$g(x)(n) = m$”. Since

$g(x)(n) \neq m \iff (\exists i \in \omega) i \neq m \land g(x)(n) = i$,

we have that the ternary relation is in $\Gamma$ iff it is in the dual $\neg \Gamma$. Since

$g(x) = y \iff (\forall n \in \omega)[(\forall m \in \omega) m = y(n) \rightarrow g(x)(n) = m]$,

if the ternary relation “$g(x)(n) = m$” is in $\Gamma$ then the binary relation “$g(x) = y$” is in $\Gamma$. Similarly, since

$g(x)(n) = m \iff (\exists y \in \omega \omega)[g(x) = y \land y(n) = m]$,

$g(x)(n) = m \iff (\forall y \in \omega \omega)[g(x) = y \Rightarrow y(n) = m]$,

if the binary relation is in $\Gamma$, then the ternary relation is in $\exists \omega \omega \Gamma$ and $\forall \omega \Gamma$. Thus, the binary relation “$g(x) = y$” is $\Sigma^1_n$ iff it is $\Pi^1_n$ iff it is $\Delta^1_n$ iff the ternary relation “$g(x)(n) = m$” is $\Sigma^1_n$ iff it is $\Pi^1_n$ iff it is $\Delta^1_n$.

Using a definition of [7], a well-ordering $\leq$ of $\omega \omega$ is called $\Gamma$-good iff it is in $\Gamma$ and whenever $P$ is a binary $\Gamma$-relation, then the relations $Q(x, y) \iff (\exists x' \leq x) P(x', y)$ and $R(x, y) \iff (\forall x' \leq x) P(x', y)$ are in $\Gamma$. Note that if $\leq$ is $\Gamma$-good, then it is also $\neg \Gamma$-good. Also, if $P$ is a binary $\Delta$-relation, then “$x$ is the $\leq$-least real such that $P(x, y)$” is also a $\Delta$-relation. If $V = L[r]$ for some $r \in \omega \omega$, then there is a $\Sigma^1_2$-good wellordering of $\omega \omega$.
In the construction to follow, for \( \alpha < \omega_1 \), we will use codes to talk about the \( c \)-th \( \Sigma^0_\alpha \) function \( h_c : \omega^\omega \to \omega^\omega \), where \( c \in \omega^\omega \).

**Lemma 2.1.** Assume CH, and so \( \text{add}(\mathcal{B}) = 2^\omega \). Assume there is a \( \Sigma^1_2 \)-good well-ordering \( \leq \) of \( \omega^\omega \). Fix \( \alpha < \omega_1 \). There is an uncountable set \( S \subseteq \omega^\omega \) along with a \( \Delta^1_2 \) function \( H : \omega^\omega \to \omega^\omega \) such that whenever \( c \in \omega^\omega \) is a code for a \( \Sigma^0_\alpha \) function \( h : \omega^\omega \to \omega^\omega \), then \( H(c) \not\in h^{\prime \prime}S \). That is, \( H \) witnesses that no \( \Sigma^0_\alpha \) function surjects \( S \) onto \( \omega^\omega \).

**Proof.** We will define a function \( I : \omega^\omega \to \omega^\omega \times \omega^\omega \) and we will have

\[
S = \{ a : I(c) = (a, y) \text{ for some } c \in \omega^\omega \}.
\]

We will also have \( H(c) = y \), where \( I(c) = (a, y) \).

We will define \( I \) from a function \( J : \omega^\omega \to \omega^\omega \). Fix \( c \in \omega^\omega \). \( J(c) \) will code a function \( F_c \) from \( \{ x : x \leq c \} \) to \( \omega^\omega \times \omega^\omega \). We will have \( I(c) = F_c(c) \). \( J(c) \) will be the \( \leq \)-good code for \( F_c \). For each \( x \leq c \), let \( (a_x, y_x) = F_c(x) \). Define \( F_c \) to be the unique function that satisfies the following for all \( x \leq c \).

1) \( y_x \) is the \( \leq \)-least real such that
   1a) \( h^{-1}_x(y_x) \) is meager;
   1b) \( \forall x' \leq x \) \( x' \neq x \Rightarrow a_{x'} \not\in h^{-1}_x(y_x) \).

2) \( a_x \) is the \( \leq \)-least real such that
   2a) \( \forall x' \leq x \) \( a_x \neq a_{x'} \);
   2b) \( \forall x' \leq x \) \( a_x \not\in h^{-1}_x(y_{x'}) \).

Since \( \text{add}(\mathcal{B}) = 2^\omega \), by the argument in Proposition [1.6] there is such an \( F_c \). One can see that this \( F_c \) is unique. We will now show that the relation \( \text{“} d \in \omega^\omega \text{ codes } F_c \text{”} \) is \( \Delta^1_2 \). It will follow that \( J \) is \( \Delta^1_2 \), and from this that \( H \) is \( \Sigma^1_2 \).

Note that the well-ordering \( \leq \) is in fact \( \Delta^1_2 \). First note that \( \text{“} d \in \omega^\omega \text{ codes } \{ x : x \leq c \} \text{”} \) is \( \Delta^1_2 \). Quantifying over the reals in the countable set coded by a \( d \) is a number quantifier, not a real quantifier. So, \( \text{“} (\forall x \in \text{ the set coded by } d) \ x \leq c \text{”} \) is \( \Delta^1_2 \). On the other hand, \( \text{“} (\forall x \leq c) \ x \in \text{ the set coded by } d \text{”} \) is \( \Delta^1_2 \) because \( \leq \) is \( \Sigma^1_2 \)-good and \( \Pi^1_2 \)-good. This shows that the relation \( \text{“} d \in \omega^\omega \text{ codes } \{ x : x \leq c \} \text{”} \) is \( \Delta^1_2 \), and similarly \( \text{“} d \in \omega^\omega \text{ codes a function from } \{ x : x \leq c \} \text{ to } \omega^\omega \times \omega^\omega \text{”} \) is \( \Delta^1_2 \).

We will now prove that \( \text{“} d \in \omega^\omega \text{ codes } F_c \text{”} \). Because \( \leq \) is \( \Sigma^1_2 \)-good and \( \Pi^1_2 \)-good, it suffices to show that 1) and 2) are \( \Delta^1_2 \). First, 1a) is certainly \( \Delta^1_2 \). Next, since \( a_{x'} \not\in h^{-1}_x(y_x) \) is \( \Delta^1_2 \) and \( \leq \) is \( \Sigma^1_2 \)-good and \( \Pi^1_2 \)-good we have that 1b) is \( \Delta^1_2 \). So, the conjunction of 1a) and 1b) is \( \Delta^1_2 \). The property of being the least real that satisfies a \( \Delta^1_2 \) relation is \( \Delta^1_2 \), so it follows that 1) is \( \Delta^1_2 \).
Now “\(a_x \neq a_y\)” is certainly \(\Delta^1_2\), so 2a) is \(\Delta^1_2\) because \(\leq\) is \(\Sigma^1_2\)-good and \(\Pi^1_2\)-good. Similarly, 2b) is \(\Delta^1_2\). Now the conjunction of 2a) and 2b) is \(\Delta^1_2\), and so 2) is \(\Delta^1_2\) as well.

\[\square\]

**Corollary 2.2.** Assume there is a \(\Sigma^1_3\)-good well-ordering \(\leq\) of \(\omega\). Let \(\{f^a : a \in \omega\}\) be any family of functions from \(\omega\) to \(\omega\) such that \((a, x) \mapsto f^a(x)\) is Borel. Then there is a \(\Delta^1_2\) function \(g : \omega \to \omega\) that is disjoint from uncountably many of the \(f^a\) functions.

**Proof.** Fix \(\alpha < \omega_1\) such that each function \(a \mapsto f^a(x)\) is \(\Sigma^0_\alpha\). Let \(S\) and \(H\) be from the Lemma above. Define \(g : \omega \to \omega\) as follows. For each \(x \in \omega\), let \(c\) be a code for the function \(a \mapsto f^a(x)\). Then \(H(c) \notin \{f^a(x) : a \in S\}\). Define \(g(x) := H(c)\). One can check that \(g\) is in fact \(\Delta^1_2\).

\[\square\]

Note that a similar argument shows that if instead there is a \(\Sigma^1_3\)-good well-ordering of \(\omega\), then there is a \(\Delta^1_3\) function disjoint from uncountably many of the \(f^a\) functions.

**Corollary 2.3.** (ZF+DC) Assume there is a family \(\{f^a : a \in \omega\}\) such that 1) \((a, x) \mapsto f^a(x)\) is Borel and 2) no \(\Delta^1_2\) function is disjoint from uncountably many of the \(f^a\) functions. Then \((\forall r \in \omega) r\) is inaccessible in \(L[r]\).

**Proof.** Fix a family \(\{f^a : a \in \omega\}\) satisfying 1). Since we are assuming ZF+DC, the statement \((\forall r \in \omega) \omega_1\) is inaccessible in \(L[r]\) is equivalent to the statement \((\forall r \in \omega) \omega_1^{L[r]} < \omega_1 \ [3]\). We will prove the contrapositive. That is, fix \(r \in \omega\) such that \(\omega_1^{L[r]} = \omega_1\). We will construct a \(\Delta^1_2\) function, with a code in \(L[r]\), that is disjoint from uncountably many of the \(f^a\) functions.

Apply the corollary above in \(L[r]\). Note that in \(L[r]\), there is a \(\Sigma^1_3\)-good well-ordering of \(\omega\). Let \(S \subseteq \omega \cap L[r]\) be uncountable in \(L[r]\) and let \(g \in L[r]\) be \(\Delta^1_2\) and disjoint, in \(L[r]\), from each \(f^a\) for \(a \in S\). Since \(S\) is uncountable in \(L[r]\) and \(\omega_1^{L[r]} = \omega_1\), we have that \(S\) is uncountable in \(V\). Let \(\tilde{g} : \omega \to \omega\) be the function in \(V\) defined by the same \(\Sigma^1_3\) and \(\Pi^1_3\) formulas that are used to define \(g\) in \(L[r]\). To see that the set defined by (one of) these formulas is indeed a function, use the Shoenfield absoluteness theorem. Now temporarily fix \(a \in S\). In \(L[r]\), the statement \(g \cap f^a = \emptyset\) is \(\Pi^1_2\). By the Shoenfield absoluteness theorem, \(\tilde{g} \cap f^a = \emptyset\). Thus, \(\tilde{g}\) is \(\Delta^1_2\) and is disjoint (in \(V\)) from each \(f^a\) for \(a \in S\).

\[\square\]
3. \( f_a \) AND \( H \)

We will use the notation from [2]: \( a, A, f_a, H, \leq, \leq^A \). That is, given a real \( a \in {}^\omega \omega \), the set \( A \subseteq \omega \) is an arbitrary set that is Turing equivalent to \( a \) and is computable from every infinite subset of itself. The function \( f_a : {}^\omega \omega \rightarrow {}^\omega \omega \) is defined as follows: let \( \eta : A \rightarrow \omega \) be a surjection such that each set \( \eta^{-1}(m) \) is infinite. The complexity of \( \eta \) does not matter. Given \( x = \langle x_0, x_1, ... \rangle \in {}^\omega \omega \), let \( i_0 < i_1 < ... \) be the indices \( i \) of which elements \( x_i \) are in \( A \). Define \( f_a(x) \in {}^\omega \omega \) to be

\[
  f_a(x) := \langle \eta(x_{i_0}), \eta(x_{i_1}), ... \rangle.
\]

To see how the coding works, consider a node \( t \in {}^<\omega \omega \). Let \( n \in \omega \) be the number of \( l \in \text{Dom}(t) \) such that \( t(l) \in A \). All \( x \in {}^\omega \omega \) that extend \( t \) agree up to the first \( n \) values of \( f_a(x) \), but not at the \( (n+1) \)-th value. By extending \( t \) by one to get \( t \triangledown k \) for some \( k \in A \), we can decide the \( (n+1) \)-th value of \( f_a(x) \) to be anything we want.

The poset \( H \), a variant of Hechler forcing, is equivalent to the forcing which consists of trees \( T \subseteq {}^<\omega \omega \) with co-finite splitting after the stem, where the ordering \( \leq \) is reverse inclusion. We present \( H \) as consisting of pairs \( (t, h) \) such that \( t \in {}^<\omega \omega \) and \( h : {}^<\omega \omega \rightarrow \omega \), where \( t \) specifies the stem and \( h \) specifies where each node beyond the stem has a final segment of successors. That is, we have \( (t', h') \leq (t, h) \) iff \( h' \geq h \) (everywhere domination), \( t' \equiv t \), and for each \( n \in \text{Dom}(t') - \text{Dom}(t) \),

\[
  t'(n) \geq h(t' \upharpoonright n).
\]

There is also a stronger ordering \( \leq^A \) defined by \( (t', h') \leq^A (t, h) \) iff \( (t', h') \leq (t, h) \) and for each \( n \in \text{Dom}(t') - \text{Dom}(t) \),

\[
  t'(n) \notin A.
\]

We will also use the main lemma from [2], which tells us a situation where we can hit a dense subset of \( H \) by making a \( \leq^A \) extension. By an \( \omega \)-model we mean a model of ZF that is possibly ill-founded but whose \( \omega \) is well-founded. Moreover, this next lemma only needs \( M \) to satisfy a fragment of ZF.

**Lemma 3.1. (Main Lemma)** Let \( M \) be an \( \omega \)-model of ZF and \( D \in \mathcal{P}^M(\mathbb{H}^M) \) a set dense\(^M \) in \( \mathbb{H}^M \). Let \( A \subseteq \omega \) be infinite and \( \Delta^1_1 \) in every infinite subset of itself, but \( A \notin M \). Then

\[
  (\forall p \in \mathbb{H}^M)(\exists p' \leq^A p) p' \in D.
\]

4. \( \text{AD}^+ \) IMPLIES \( \Psi \)

The proof of the theorem of this section similar to that of Fact [1.1]
**Definition 4.1.** A set $X \subseteq \omega^\omega$ is $\infty$-Borel iff there is a pair $(C, \varphi)$, called an $\infty$-Borel code, such that $C$ is a set of ordinals and $\varphi$ is a formula such that

$$X = \{ x \in \omega^\omega : L[C, x] \models \varphi(C, x) \}.$$ 

A similar definition applies to relations $R \subseteq \omega^\omega \times \cdots \times \omega^\omega$. We abuse language and call a set $C \subseteq \text{Ord}$ an $\infty$-Borel code for $X \subseteq \omega^\omega$ iff there is a formula $\varphi$ such that $(C, \varphi)$ is an $\infty$-Borel code for $X$.

We do not define a function $g : \omega^\omega \to \omega^\omega$ to be $\infty$-Borel iff its graph is $\infty$-Borel: if $C$ is an $\infty$-Borel code for the graph of $g : \omega^\omega \to \omega^\omega$, there is no guarantee that $g(x) \in L[C, x]$. This is the reason for the following definition:

**Definition 4.2.** A function $g : \omega^\omega \to \omega^\omega$ is $\infty$-Borel iff there is a pair $(C, \varphi)$, called an $\infty$-Borel code, such that for all $x \in \omega^\omega$ and $n, m \in \omega$,

$$g(x)(n) = m \iff L[C, x] \models \varphi(C, x, n, m).$$

We abuse language and call $C \subseteq \text{Ord}$ an $\infty$-Borel code for $g : \omega^\omega \to \omega^\omega$ iff there is a formula $\varphi$ such that $(C, \varphi)$ is an $\infty$-Borel code for $g$.

We similarly define $\infty$-Borel codes for functions $g : \omega^\omega \to \omega^\omega \times [\omega]^\omega$, etc. We will sometimes be loose and write a code $(C, \varphi)$ for the graph of $g$, but we will always mean the more technical definition. Note that if $g : \omega^\omega \to \omega^\omega$ is $\infty$-Borel with code $C$, then $g(x) \in L[C, x]$ for all $x$. Our strong definition of a function being $\infty$-Borel is justified because if every $A \subseteq \omega^\omega$ is $\infty$-Borel, then every $g : \omega^\omega \to \omega^\omega$ is $\infty$-Borel.

**Lemma 4.3.** (ZF) Assume there is no injection from $\omega_1$ into $\omega^\omega$. Let $M$ be an inner model of ZFC. Then $\mathcal{P}^M(\mathbb{H}^M)$ is countable.

**Proof.** The set $\omega^\omega \cap M$ must be countable. Every element of $\mathbb{H}^M$ corresponds to an element of $\omega^\omega \cap M$. Hence, in $V$ there is a bijection from $\mathbb{H}^M$ to $\omega$, so in $V$ there is a bijection $\eta_2 : \mathcal{P}(\mathbb{H}^M) \to \mathcal{P}(\omega)$. Since $M$ satisfies the Axiom of Choice, let $\lambda$ be the smallest ordinal such that there exists a bijection $\eta_1 : \lambda \to \mathcal{P}^M(\mathbb{H}^M)$ in $M$. We now have that $\eta_2 \circ \eta_1 : \lambda \to \mathcal{P}(\omega)$ is an injection, so by the hypothesis it must be that $\lambda < \omega_1$. Hence, $\mathcal{P}^M(\mathbb{H}^M)$ is countable. \hfill $\Box$

**Theorem 4.4.** (ZF) Assume there is no injection from $\omega_1$ into $\omega^\omega$. Let $g : \omega^\omega \to \omega^\omega$ be $\infty$-Borel with code $C \subseteq \text{Ord}$. Then for all $a \in \omega^\omega$,

$$f_a \cap g = \emptyset \Rightarrow a \in L[C].$$

Hence, $\Psi(g)$ holds.
Proof. Let $\varphi$ be such that $(C, \varphi)$ is an $\infty$-Borel code for $g$. Assume $a \notin L[C]$. We must construct an $x \in \omega^\omega$ such that $f_a(x) = g(x)$. Since $a$ and $A$ are Turing equivalent, we have $A \notin L[C]$, which allows us to apply Lemma 3.1, the main lemma. By the lemma above, $\mathcal{P}^L[C](\mathbb{H}[L[C]])$ is countable, so fix an enumeration $\langle D_n \in \mathcal{P}^L[C](\mathbb{H}[L[C]) : n \in \omega \rangle$ of the dense subsets of $\mathbb{H}[L[C]]$ in $L[C]$.

We will construct a generic $G$ for $\mathbb{H}[L[C]]$ over $L[C]$. Let $x = \bigcup \{ t : (\exists h) (t, h) \in G \}$. Let $\dot{x}$ be the canonical name for $x$. The forcing extension will be $L[C, G] = L[C, x]$.

First, apply Lemma 3.1 to get $p_0 \leq^A 1$ such that $p_0 \in D_0$. Next, apply Lemma 3.1 to get $p'_0 \leq^A p_0$ and $m_0 \in \omega$ such that

$$p'_0 \models \varphi(\dot{C}, \dot{x}, 0, \dot{m}_0).$$

Next, extend the stem of $p'_0$ by one to get $p''_0 \leq p'_0$ in a way to ensure that $f_a(x)(0) = m_0$.

Next, apply Lemma 3.1 to get $p_1 \leq^A p''_0$ such that $p_1 \in D_1$. Next, apply Lemma 3.1 to get $p'_1 \leq^A p_1$ and $m_1 \in \omega$ such that

$$p'_1 \models \varphi(\dot{C}, \dot{x}, 1, \dot{m}_1).$$

Next, extend the stem of $p'_1$ by one to get $p''_1 \leq p'_1$ in a way to ensure that $f_a(x)(1) = m_1$.

Continue like this infinitely. Since we have constructed a generic over $L[C]$, we have that for each $i < \omega$,

$$L[C, x] \models \varphi(C, x, i, m_i).$$

Since $(C, \varphi)$ witnesses that $g$ is $\infty$-Borel, this means that for each $i < \omega$,

$$g(x)(i) = m_i.$$

On the other hand, we have ensured that for each $i < \omega$,

$$f_a(x)(i) = m_i.$$ 

Thus, $f_a(x) = g(x)$, which is what we wanted to show. □

$\text{AD}^+$ is an axiom which implies $\text{AD}$, the Axiom of Determinacy, and it is open whether $\text{AD}$ implies $\text{AD}^+$.

**Corollary 4.5.** $(ZF + AD^)$ For all $g : \omega^\omega \rightarrow \omega^\omega$, $\Psi(g)$ holds.

**Proof.** $\text{AD}^+$ implies that every set of reals is $\infty$-Borel, and hence that every $g : \omega^\omega \rightarrow \omega^\omega$ is $\infty$-Borel. Also $\text{AD}^+$ implies $\text{AD}$, which in turn implies there is no injection of $\omega_1$ into $\omega^\omega$. □
5. More on $\Delta_1^2$ functions

It is well-known that $\Sigma^2_2$ has the PSP iff every $c \in \omega\omega$ is inaccessible in $L[c]$. We have a similar result:

**Corollary 5.1.** The following are equivalent:

1) $(\forall g \in \Delta_1^2) \Psi(g)$.
2) $(\forall c \in \omega\omega) \omega_1$ is inaccessible in $L[c]$.

The 1) implies 2) direction follows from Corollary 2.3. The 2) implies 1) direction is the second case in Fact 1.1. Specifically, it was shown that if $\omega_1$ is inaccessible in $L[c]$, then for all $\Delta_1^2(c)$ functions $g$ and for all $a \in \omega\omega$, if $g \cap f_a = \emptyset$ then $a \in L[c]$. The assumption that $\omega_1$ is inaccessible in $L[c]$ is only needed to get $\mathcal{P}^{L[c]}(\mathbb{H}_{L[c]})$ to be countable. If this set is not countable, we can force it to be countable and then use the Shoenfield absoluteness theorem. Note that in the following result, $\omega_1 \cap L[c]$ need not be countable; indeed $\{a : g \cap f_a = \emptyset\}$ need not be countable.

**Proposition 5.2.** Fix $c \in \omega\omega$. Let $g : \omega\omega \to \omega \omega$ be a $\Delta_1^2(c)$ function. Fix $a \in \omega\omega$. Then

$$g \cap f_a = \emptyset \Rightarrow a \in L[c].$$

**Proof.** Assume $a \notin L[c]$. We will show $g \cap f_a \neq \emptyset$. Let $\varphi$ be a $\Sigma^1_2$ formula such that for all $x \in \omega\omega$ and $n, m \in \omega$,

$$g(x)(n) = m \iff \varphi(c, x, n, m).$$

By the Shoenfield absoluteness theorem, $\varphi$ is absolute between all inner models. Now let $G$ be generic over $V$ to collapse $\mathcal{P}^{L[c]}(\mathbb{H}_{L[c]})$ to be countable. In $V[G]$, perform the construction done in Theorem 4.4. That is, get a real $x \in (\omega\omega)^{V[G]}$ and a sequence $\langle m_i \in \omega : i < \omega \rangle \in V[G]$ such that for all $i < \omega$, $L[c, x] \models \varphi(c, x, i, m_i)$ and $f_a(x)(i) = m_i$. Thus, $\tilde{g}(x) = f_a(x)$ holds in $V[G]$, where $\tilde{g}$ is the function defined in $V[G]$ using the formula $\varphi$ with the parameter $c$. Hence, $(\exists z) \tilde{g}(z) = f_a(z)$ holds in $V[G]$, and this is a $\Sigma^1_2(c,a)$ statement, so it holds in $V$. That is, $g \cap f_a \neq \emptyset$ holds in $V$. 

**Corollary 5.3.** Fix $c \in \omega\omega$. If $\omega\omega \cap L[c]$ is countable, then $\Psi(g)$ holds for every $\Delta_1^2(c)$ function $g$.

This last corollary is analogous to the situation with the PSP. Fix $c \in \omega\omega$. By the Mansfield-Solovay theorem [8], if $A \subseteq \omega\omega$ is $\Sigma^1_2(c)$ and $A \not\subseteq L[c]$, then $A$ has a perfect subset. Thus, if $\omega\omega \cap L[c]$ is countable, then $\Sigma^1_2(c)$ satisfies the PSP.
5.1. $\Delta^1_3$ Functions. Suppose there is a measurable cardinal. Then $(\forall r \in \omega \omega) \omega_1$ is inaccessible in $L[r]$. Thus, by Corollary 5.1, we have $(\forall g \in \Delta^1_3) \Psi(g)$. One may wonder if the existence of a measurable cardinal also proves $(\forall g \in \Delta^1_3) \Psi(g)$. That answer is no because if $V = L[U]$ for some normal ultrafilter $U$, then there is a $\Sigma^1_3$-good well-ordering of $\omega \omega$. Hence, by the comments following Corollary 2.2 there is a $\Delta^1_3$ function $g$ such that $\neg \Psi(g)$.

6. Consistency Strength Upper Bound

In this section we will show that $(\forall g) \Psi(g)$ holds in the Solovay model. This establishes that the following theories are equiconsistent:

- $\text{ZFC} + \exists$ inaccessible cardinal;
- $\text{ZF} + \text{DC} + (\forall g) \Psi(g)$;
- $\text{ZF} + \text{DC} + \Psi$;
- $\text{ZF} + \text{DC} + \Psi$ for only $\Delta^1_3$'s.

**Theorem 6.1.** Let $M$ be an inner model of $\text{ZFC}$ and let $\kappa$ be a strongly inaccessible cardinal in $M$. Assume $V = M[G]$ where $G$ is generic for the Levy collapse of $\kappa$ over $M$. Fix $C \in \omega \text{Ord}$ and let $g : \omega \omega \to \omega \omega$ be such that there is a formula $\varphi$ such that for each $x \in \omega \omega$ and $n, m \in \omega$, $g(x)(n) = m \iff \varphi(C, x, n, m)$.

Then for all $a \in \omega \omega$, $f_a \cap g = \emptyset \Rightarrow a \in M[C]$.

**Proof.** Given any $x \in \omega \omega$, by the factoring of the Levy collapse for countable sets of ordinals, $V$ is generic over $M[C, x]$ by the Levy collapse of $\kappa$, and $\omega_1 = \kappa$ is inaccessible in $M[C, x]$. Since the Levy collapse is homogeneous, for any $x, n, m$ we have

$$\varphi(C, x, n, m) \iff M[C, x] \models 1 \models \varphi(\bar{C}, \bar{x}, \bar{n}, \bar{m}).$$

Letting $\bar{\varphi}(C, x, n, m)$ be the formula $1 \models \varphi(\bar{C}, \bar{x}, \bar{n}, \bar{m})$, we have

$$g(x)(n) = m \iff M[C, x] \models \bar{\varphi}(C, x, n, m).$$

Hence, each model $M[C, x]$ can compute the value of $g(x)$. Now start with $M[C]$ and assume $a \notin M[C]$. Note that $P^{M[C]}(\mathbb{H}^M[C])$ is countable, because $\omega_1 = \kappa$ is inaccessible in $M[C]$. We can construct $x \in \omega \omega$ to be a generic for $\mathbb{H}^M[C]$ over $M[C]$ in the special way, as was done in Theorem 4.4, to get $f_a(x) = g(x)$. This finishes the theorem. 

**Corollary 6.2.** Let $\kappa$ be an inaccessible cardinal. Let $G$ be generic for the Levy collapse of $\kappa$ over $V$. Then

$$\text{HOD}(\omega \text{Ord})^{V[G]} \models (\forall g) \Psi(g).$$
7. Functions in \( L(\mathbb{R})[\mathcal{U}] \)

The point of this section is to show that functions \( g : \omega \omega \rightarrow \omega \omega \) in models of the form \( L(\mathbb{R})[\mathcal{U}] \), where \( \mathcal{U} \) is a selective ultrafilter on \( \omega \), satisfy \( \Psi(g) \). Hence, the existence of a non-principal ultrafilter on \( \omega \) is not enough to imply \( \neg \Psi \).

Note that the next lemma applies to the forcing over \( L(\mathbb{R}) \) to add a Cohen subset of \( \omega_1 \). However, in that forcing extension, there is a well-ordering of \( \mathbb{R} \), so \( \Psi \) fails there. The significance of the lemma is that if PSP holds in the extension, then \( (\forall g) \Psi(g) \) must also hold there.

In this next lemma, given \( \dot{g} \in L(\mathbb{R}) \), we need to uniformize a certain binary relation associated to \( \dot{g} \). Since a uniformization may not exist in \( L(\mathbb{R}) \), we choose to work in a model that can uniformize every binary relation on \( \omega \omega \) in \( L(\mathbb{R}) \). Alternatively we could prove the weaker result, assuming \( \mathcal{Q} \) is relatively low in the Wadge hierarchy, that for every \( \dot{g} \in (\Sigma^2_1)^{L(\mathbb{R})} \) the following statement \( P(\dot{g}) \) holds: for every \( q \in \mathcal{Q} \), there is a countable set \( C_{g,q} \subseteq \omega \omega \) such that \( (\forall a \in \omega \omega) (q \vert_{\mathcal{Q}} \dot{g} \cap \check{f}_a = \emptyset)^{L(\mathbb{R})} \Rightarrow a \in C_{g,q} \). Then we could use \( \Sigma^2_1 \) reflection in \( L(\mathbb{R}) \) to get that \( P(\dot{g}) \) holds for all \( \dot{g} \in L(\mathbb{R}) \). See [5] for more on \( \Sigma^2_1 \) reflection.

**Lemma 7.1.** Assume AD\(^+\) and that every binary relation on \( \omega \omega \) in \( L(\mathbb{R}) \) can be uniformized. Let \( \mathcal{Q} \in L(\mathbb{R}) \) be a forcing that does not add reals and whose underlying set is \( \omega \omega \). Let \( \dot{g} \in L(\mathbb{R}) \) be such that \( (1 \vert_{\mathcal{Q}} \dot{g} : \omega \omega \rightarrow \omega \omega)^{L(\mathbb{R})} \). Then there exists a set of ordinals \( C \subseteq \text{Ord} \) such that \( (\forall q \in \mathcal{Q}) (\forall a \in \omega \omega) \)

\[
(q \vert_{\mathcal{Q}} \dot{g} \cap \check{f}_a = \emptyset)^{L(\mathbb{R})} \Rightarrow a \in L[C, q].
\]

**Proof.** Since we can uniformize every binary relation on \( \omega \omega \) that is in \( L(\mathbb{R}) \), let \( u : \mathcal{Q} \times \omega \omega \rightarrow \mathcal{Q} \times \omega \omega \) be such that \( (\forall q \in \mathcal{Q}) (\forall x \in \omega \omega) \), if \( u(q, x) = (q', y) \), then \( q' \leq q \) and

\[
(q' \vert_{\mathcal{Q}} \dot{g}(\check{x}) = \dot{y})^{L(\mathbb{R})}.
\]

Let \( (C, \varphi) \) be an \( \infty \)-Borel code for \( u \). That is, \( (\forall q, q' \in \mathcal{Q}) (\forall x, y \in \omega \omega) u(q, x) = (q', y) \Leftrightarrow L[C, q, x, q', y] \models \varphi(C, q, x, q', y) \).

Note that by our convention for \( \infty \)-Borel codes for functions to \( \omega \omega \) or similar ranges, if \( u(q, x) = (q', y) \), then \( q', y \in L[C, q, x] \).

Now fix \( q \in \mathcal{Q} \). Assume that \( a \notin L[C, q] \). We will show that \( \neg (q \vert_{\mathcal{Q}} \dot{g} \cap \check{f}_a = \emptyset)^{L(\mathbb{R})} \). We will do this by constructing a \( q' \leq q \) and an \( x \in \omega \omega \) such that \( (q' \vert_{\mathcal{Q}} \dot{g}(\check{x}) = \check{f}_a(\check{x}))^{L(\mathbb{R})} \). Consider \( L[C, q] \). The \( x \) will be generic over this model by the forcing \( \mathbb{H}^{L[C, q]} \). Then, setting \( (q', y) = u(q, x) \), we will have \( (q' \vert_{\mathcal{Q}} \dot{g}(\check{x}) = \dot{y})^{L(\mathbb{R})} \). At the same time, we will construct \( x \) so that \( f_a(x) = y \).
Let \( \dot{x} \) be such that \( 1 \models \dot{x} = \bigcup \{ t : (\exists h) (t, h) \in \dot{G} \} \), where \( \dot{G} \) is the canonical name for the generic filter. That is, \( \dot{x} \) is a name for the real \( x \) we will construct. We will now construct \( x \) by building a generic filter for \( \mathbb{H}^{L[C,q]} \) over \( L[C,q] \). Let \( \dot{y}, \dot{y} \in L[C,q] \) be such that

\[
(1 \models_{\mathbb{H}} \varphi(\dot{C}, \dot{q}, \dot{x}, \dot{q}', \dot{y}))^{L[C,q]}.
\]

Then, letting \( q' = (\dot{q}')_x \) and \( y = (\dot{y})_x \) be the valuations of these names with respect to the generic \( x \), we will have \( L[C, q, x] \models \varphi(C, q, x, q', y) \), so \( u(q, x) = (q', y) \), which implies \( q' \subseteq q \) and \( q' \models_{\mathbb{Q}} (\dot{g}(\dot{x}) = \dot{y})^{L(R)} \).

Let \( \langle D_i : i < \omega \rangle \) be an enumeration of the dense subsets of \( \mathbb{H}^{L[C,q]} \) in \( L[C,q] \). Let \( p_0 \subseteq^A 1 \) be in \( D_0 \). Let \( p_0' \subseteq^A p_0 \) and \( m_0 \in \omega \) be such that \( p_0' \) decides \( \dot{y}(0) = m_0 \). That is, \( (p_0' \models_{\mathbb{H}} \dot{y}(0) = m_0)^{L[C,q]} \). Let \( p_0'' \subseteq p_0' \) extend the stem of \( p_0' \) by one to ensure that \( f_a(x)(0) = m_0 \).

Now let \( p_1 \subseteq^A p_0'' \) be in \( D_1 \). Let \( p_1' \subseteq^A p_1 \) and \( m_1 \in \omega \) be such that \( (p_1' \models_{\mathbb{H}} \dot{y}(1) = m_1)^{L[C,q]} \). Let \( p_1'' \subseteq p_1' \) extend the stem of \( p_1' \) by one to ensure that \( f_a(x)(1) = m_1 \).

Continue this procedure infinitely. The descending sequence of conditions constructed yields a generic ultrafilter \( G \) for \( \mathbb{H}^{L[C,q]} \). By the way \( x = (\dot{x})_G \) was constructed, we have \( f_a(x) = m_i \) for all \( i < \omega \). We also have \( y(i) = m_i \) for all \( i < \omega \). Finally, we have that \( (q' \models_{\mathbb{Q}} \dot{g}(\dot{x}) = \dot{y})^{L(R)} \). This completes the proof.

**Observation 7.2.** Assume that PSP holds. Then a forcing extension that does not add reals satisfies PSP iff every uncountably set of reals in the extension has an uncountable subset in the ground model. This is because every perfect set of reals in the extension is already in the ground model.

Paul Larson pointed out this next argument, along with using the generic absoluteness of the theory of \( L(\mathbb{R}) \).

**Lemma 7.3.** Assume \( \omega_1 < t \). Let \( \mathbb{Q} \) be the \( P(\omega)/\text{Fin} \) forcing. Then \( (1 \models_{\mathbb{Q}} \text{PSP})^{L(\mathbb{R})} \).

**Proof.** Fix \( \dot{S} \in L(\mathbb{R}) \) and \( q \) such that \( q \models \dot{S} \subseteq \omega \) is uncountable\(^{L(\mathbb{R})} \). By induction, construct a sequence \( \langle (q_\alpha, b_\alpha) : \alpha < \omega_1 \rangle \) such that 1) the \( b_\alpha \)'s are distinct reals, 2) the \( q_\alpha \)'s are decreasing with \( q \geq q_0 \), and 3) \( (q_\alpha \models_{\mathbb{Q}} \dot{b}_\alpha \in \dot{S})^{L(\mathbb{R})} \) for each \( \alpha < \omega_1 \). One does not get stuck at limit stages because \( P(\omega)/\text{Fin} \) is countably closed. Let \( q' \) be a lowerbound of the \( q_\alpha \)'s, which exists because they form a decreasing, with respect to almost inclusion, sequence of infinite subsets of \( \omega \), and this sequence cannot be maximal because \( \omega_1 < t \). We have \( q' \models_{\mathbb{Q}} \{ \dot{b}_\alpha : \alpha < \omega_1 \} \) is an uncountable subset of \( \dot{S} \).  

\( \square \)
Theorem 7.4. Assume there is a proper class of Woodin cardinals. Let $U$ be a selective ultrafilter on $\omega$. Let $g : \omega^\omega \to \omega^\omega$ be in $L(\mathbb{R})[U]$. Then $\Psi(g)$.

Proof. Let $Q$ be the $P(\omega)/\text{Fin}$ forcing. Since there is a proper class of Woodin cardinals, the first order theory of $L(\mathbb{R})$ cannot be changed by any set sized forcing. There is a forcing extension of $V$ in which $\omega_1 < t$. By Lemma 7.3, in that forcing extension we have $(1 \Vdash_Q \text{PSP})^{L(\mathbb{R})}$. Thus, in $V$ we have $(1 \Vdash_Q \text{PSP})^{L(\mathbb{R})}$.

Another consequence of a proper class of Woodin cardinals is that an ultrafilter on $\omega$ is selective iff it is $Q$-generic over $L(\mathbb{R})$ (see [1] and [4]). Thus, we will show that every name $\dot{g} \in L(\mathbb{R})$ for a function from $\omega^\omega$ to $\omega^\omega$ satisfies $L(\mathbb{R}) \models 1 \Vdash_Q \dot{g}$.

Towards a contradiction, fix $\dot{g} \in L(\mathbb{R})$ and $q \in Q$ such that $L(\mathbb{R}) \models q \Vdash_Q \dot{g} \cap \check{f}_a = \emptyset$ is uncountable. Since $L(\mathbb{R}) \models q \Vdash_Q \text{PSP}$, by the observation above fix a condition $q' \leq q$ and an uncountable set $S \subseteq \omega^\omega$ in $L(\mathbb{R})$ such that for all $a \in S$, $L(\mathbb{R}) \models q' \Vdash_Q \dot{g} \cap \check{f}_a = \emptyset$.

Again using the assumption of a proper class of Woodin cardinals, fix an inner model $M \supseteq L(\mathbb{R})$ of AD$^+$ that contains a uniformization for every binary relation on $\omega^\omega$ that is in $L(\mathbb{R})$. Apply Lemma 7.1 in $M$ to get $C \subseteq \text{Ord}$. We have $(\forall a \in S) a \in L[C, q']$, which is a contradiction because since $L[C, q']$ is an inner model of ZFC in the model $M$ of AD, $\omega^\omega \cap L[C, q']$ is countable. \qed

8. Final Remarks

It is natural to think that $(\forall g \in \Gamma) \Psi(g)$ is somehow related to a pointclass having a largest countable set. Let $\Gamma$ be an adequate pointclass closed under quantifying over $\omega$ and assume $(\forall g \in \Gamma) \Psi(g)$. Let $\Gamma' := \forall^\omega \Gamma$. For each $g : \omega^\omega \to \omega^\omega$, let $D_g := \{a \in \omega^\omega : g \cap f_a = \emptyset\}$.

Each $D_g$ is countable and in $\Gamma'$. Suppose the union $C_{\Gamma'}$ of all countable sets $D \subseteq \omega^\omega$ in $\Gamma'$ is countable. We have $(\forall g \in \Gamma)(\forall a \in \omega^\omega) g \cap f_a = \emptyset \Rightarrow a \in C_{\Gamma'}$.

However, only knowing that $C_{\Gamma'}$ is countable does not seem to give us a way to prove that $(\forall g \in \Gamma) \Psi(g)$. We close with two questions.
Question 8.1. What is the relationship between \((\forall g) \Psi(g)\) and other regularity properties?

In all models where we have shown that \((\forall g) \Psi(g)\) holds, we also know that the PSP and Ramsey properties hold. In particular, all sets of reals in \(L(\mathbb{R})[\mathcal{U}]\) are Ramsey and satisfy the perfect set property.

Question 8.2. Does \(\text{AD}\) imply \((\forall g) \Psi(g)\)?

We have shown that \(\text{AD}^+\) suffices. This is analogous to the situation for the Ramsey property: \(\text{AD}^+\) implies that all sets of reals are Ramsey, but it is unknown whether \(\text{AD}\) alone implies this.

9. Acknowledgements

I would like to thank Andreas Blass, Trevor Wilson, and Paul Larson for discussions on this project. Larson pointed out the arguments for Proposition 1.6 and Lemma 7.3. He also verified that a proper class of Woodins implies every relation on \(\omega\) in \(L(\mathbb{R})\) can be uniformized in some model of \(\text{AD}^+\). Wilson explained how much truth small forcings of \(\mathcal{M}_n(c)\) can compute. He also explained how \(\text{PD}\) suffices for both Fact 1.1 and Fact 1.2 instead of \(\omega\) Woodin cardinals.

References


Mathematics Department, University of Denver, Denver, CO 80208, U.S.A.

E-mail address: Daniel.Hathaway@du.edu