The Halpern-Läuchli Theorem at a Measurable Cardinal

Dan Hathaway, joint with Natasha Dobrinen

University of Denver

Daniel.Hathaway@du.edu

July 20, 2017
Definition

Let $\kappa$ be a regular cardinal. A tree $T \subseteq <\kappa \kappa$ is regular iff it is

1) perfect,

2) suitable (every maximal branch has length $\kappa$), and

3) a $\kappa$-tree (every level $T(\alpha) := T \cap \alpha \kappa$ of $T$ has size $< \kappa$).

Note: If $\kappa$ is not strongly inaccessible, there are no regular trees.

Definition

Given sets $T_0, \ldots, T_{d-1} \subseteq <\kappa \kappa$, $T_0 \otimes \ldots \otimes T_{d-1}$ is the set of $d$-tuples $\langle t_0, \ldots, t_{d-1} \rangle$ such that each $t_i \in T_i$ and the $t_i$'s are all on the same level.

Definition

Given sets $A, X \subseteq <\kappa \kappa$, we say that $X$ dominates $A$ iff each $a \in A$ is extended by some $x \in X$. 
SDHL stands for “Somewhere Dense Halpern-Läuchli”.

**Definition**

Given a cardinal $\sigma > 0$, $\text{SDHL}(d, \sigma, \kappa)$ is the statement that given any regular trees $T_0, ..., T_{d-1} \subseteq <\kappa \kappa$ and any level coloring $c : T_0 \otimes ... \otimes T_{d-1} \to \sigma$, there are levels $l < l' < \kappa$, a sequence of nodes $\langle t_i \in T_i(l) : i < d \rangle$, and a sequence of sets $\langle X_i \subseteq T_i(l') : i < d \rangle$ such that each $X_i$ dominates $\text{Succ}_{T_i}(t_i)$ and $c$ is constant on $X_0 \otimes ... \otimes X_{d-1}$. 
**Definitions: HL**

**HL** stands for “Strong Subtree Halpern-Läuchli”, and **HL**(d, σ, κ) is equivalent to **SDHL**(d, σ, κ) when κ is strongly inaccessible.

**Definition**

Given regular trees S ⊆ T ⊆ <κκ, we say that S is a **strong** subtree of T as witnessed by A ∈ [κ]κ iff for each l ∈ κ and t ∈ S(l),

1) If l /∈ A, then |Succ_S(t)| = 1;

2) If l ∈ A, then Succ_S(t) = Succ_T(t).

**Definition**

Given a cardinal σ > 0, **HL**(d, σ, κ) is the statement that given any regular trees T₀, ..., T_{d−1} ⊆ <κκ and any level coloring c : T₀ ⊗ ... ⊗ T_{d−1} → σ, there are strong trees S₀ ⊆ T₀, ..., S_{d−1} ⊆ T_{d−1} all witnessed by the same set of levels A ∈ [κ]κ and (∀l ∈ A) c is constant on S₀(l) ⊗ ... ⊗ S_{d−1}(l).

For the rest of this presentation, assume 0 < d < ω and 0 < σ < κ.
Complexity and reflection at a measurable

SDHL\((d, \sigma, \kappa)\) is a \(\Pi_1\) statement about \(V_{\kappa+1}\). Let \(M\) be a model of ZF such that \(V_\kappa \subseteq M\). If SDHL\((d, \sigma, \kappa)\) is true in \(V\), then it is true in \(M\). If \(V_{\kappa+1} \subseteq M\), then the other direction holds.

HL\((d, \sigma, \kappa)\) is a \(\Pi_2\) statement about \(V_{\kappa+1}\).

**Proposition (D., H.)**

Let \(\kappa\) be a measurable cardinal with a normal measure \(U\). Fix \(d\) and \(\sigma < \kappa\). Then SDHL\((d, \sigma, \kappa)\) iff

\[
\{\alpha < \kappa : \text{SDHL}(d, \sigma, \alpha)\} \in U.
\]

The same is true for HL\((d, \sigma, \kappa)\) in place of SDHL.

Proof: Let \(j : V \rightarrow M\) be the ultrapower embedding. Because \(V_{\kappa+1} \subseteq M\), SDHL\((d, \sigma, \kappa)\) \(\iff\) SDHL\((d, \sigma, \kappa)^M\). By Łos’s Theorem, SDHL\((d, \sigma, \kappa)^M\) \(\iff\) \(\{\alpha < \kappa : \text{SDHL}(d, \sigma, \alpha)\} \in U\). The same argument works for HL\((d, \sigma, \kappa)\) in place of SDHL.
Proposition (D., H.)

Assume that

\[ S := \{ \alpha < \kappa : \text{SDHL}(d, \sigma, \alpha) \} \]

is stationary. Then SDHL\((d, \sigma, \kappa)\) holds.

Let \( \langle T_i \subseteq \kappa : i < d \rangle \) be a sequence of regular trees and let \( c : \bigotimes_{i<d} T_i \rightarrow \sigma \) be a coloring. If we can find an \( \alpha < \kappa \) such that each \( T_i \cap \alpha < \kappa \) is an \( \alpha \)-tree and SDHL\((d, \sigma, \alpha)\) holds, then we will be done. An elementary argument shows that for each \( i < d \), there is a club \( C_i \subseteq \kappa \) such that \( (\forall \alpha \in C_i) \ T_i \cap \alpha < \kappa \) is an \( \alpha \)-tree. The set \( \bigcap_{i<d} C_i \) is a club, so it must intersect \( S \). An \( \alpha < \kappa \) in the intersection is as desired.

Corollary

If SDHL\((d, \sigma, \alpha)\) holds for a stationary set of \( \alpha < \kappa \), then SDHL\((d, \sigma, \kappa)\) holds in \( V \) and in any \( \leq \kappa \)-closed forcing extension.
Proving HL

\( \text{HL}(d, \sigma, \omega) \) can be proved by induction on \( d < \omega \) (see [7]). The successor step involves a fusion argument. This cannot be generalized to the \( \kappa > \omega \) case because the intersection of a decreasing sequence of regular trees may not be regular.

There is another proof of \( \text{HL}(d, \sigma, \omega) \) (see [3]) which adds many Cohen reals by forcing, and uses an ultrafilter in the extension to make selections. This generalizes to the \( \kappa > \omega \) case if we assume that \( \kappa \) is measurable in the extension:

**Theorem (see [1])**

Let \( \lambda > \kappa \) satisfy \( \lambda \rightarrow (\kappa)^d_{\kappa} \). Assume \( \kappa \) is measurable in the forcing extension where we add \( \lambda \) many Cohen subsets of \( \kappa \). Then \( \text{HL}(d, \sigma, \kappa) \) holds (in the ground model).

In [6], there is a theorem with a similar hypothesis and the conclusion implies \( \text{HL}(1, \sigma, \kappa) \) for all \( \sigma < \kappa \), not just finite \( \sigma < \omega \) which we mentioned was true before.
The cardinal $\kappa$ is $\alpha$-strong iff there is an elementry embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \kappa$ and $V_{\kappa+\alpha} \subseteq M$. By a (slight modification of a) theorem of Woodin (see [4] for a proof), if GCH holds and $\kappa$ is $(\kappa + d)$-strong, then there is a forcing extension in which $\kappa$ is measurable and remains measurable after adding $\lambda = \kappa^+ + d$ Cohen reals. This gives us the following:

**Corollary**

Assume there is a model in which GCH holds and there is a cardinal $\kappa$ which is $(\kappa + d)$-strong. Then there is a forcing extension in which $\kappa$ is measurable and $\text{HL}(d, \sigma, \kappa)$ holds.

Question: is the existence of a $(\kappa + d)$-strong cardinal equiconsistent with there existing a measurable $\kappa$ such that $(\forall \sigma < \kappa) \text{ HL}(d, \sigma, \kappa)$?
**Preservation by small forcings**

The following works for $\text{HL}$ in place of $\text{SDHL}$.

**Theorem (D., H.)**

Let $\mathbb{P}$ be a forcing of size $< \kappa$. Then $\text{SDHL}(d, \sigma \cdot |\mathbb{P}|, \kappa)$ implies $1 \models_{\mathbb{P}} \text{SDHL}(d, \sigma, \kappa)$.

**Sketch ($d = 2$ case):** Given a name $\dot{T}$ for a regular tree, let $\text{Der}(\dot{T})$ be the set of all equivalence classes of pairs $(\dot{\tau}, \alpha)$ such that

$$1 \models_{\mathbb{P}} (\dot{\tau} \in \dot{T} \text{ and } \text{Length}(\dot{\tau}) = \check{\alpha}),$$

where $(\dot{\tau}_1, \alpha_1) \equiv (\dot{\tau}_2, \alpha_2)$ iff $1 \models_{\mathbb{P}} (\dot{\tau}_1 = \dot{\tau}_2)$. Order $\text{Der}(\dot{T})$ by $[(\dot{\tau}_1, \alpha_1)] \leq [(\dot{\tau}_2, \alpha_2)]$ iff $1 \models_{\mathbb{P}} \dot{\tau}_1 \subseteq \dot{\tau}_2$. Fact: $\text{Der}(\dot{T})$ is a regular tree.

Given names $\dot{T}_1, \dot{T}_2$ for regular trees and a name $\dot{c}$ such that $1 \models_{\mathbb{P}} [\dot{c} : \dot{T}_1 \otimes \dot{T}_2 \to \check{\sigma}]$, let $c : \text{Der}(\dot{T}_1) \otimes \text{Der}(\dot{T}_2) \to \mathbb{P} \times \sigma$ be any coloring such that for each $r = \langle (\dot{\tau}_1, \alpha), (\dot{\tau}_2, \alpha) \rangle$,

$$\text{First}(c(r)) \models_{\mathbb{P}} \dot{c}(\dot{\tau}_1, \dot{\tau}_2) = \text{Second}(c(r)).$$
Corollary

If GCH holds and \( \kappa \) is \((\kappa + d)\)-strong, then there is a forcing extension in which \((\forall \sigma < \kappa)\) SDHL\((d, \sigma, \kappa)\) holds, but \( \kappa \) is not weakly compact.

Proof: First force over \( V \) to get a model \( V[G_1] \) in which SDHL holds at \( \kappa \), which is also measurable. By a theorem of Hampkins, any non-trivial forcing of size \( < \kappa \) followed by a non-trivial \( < \kappa \)-closed forcing will make \( \kappa \) NOT weakly compact. Perform any non-trivial forcing of size \( < \kappa \) over \( V[G_1] \) to get \( V[G_1][G_2] \). This will preserve SDHL at \( \kappa \) by the previous theorem. Since \( \kappa \) is measurable in \( V[G_1][G_2] \), SDHL holds on a stationary (in fact, measure one) subset of \( \kappa \). Now perform any non-trivial \( < \kappa \)-closed forcing over \( V[G_1][G_2] \) to get \( V[G_1][G_2][G_3] \). Stationary subsets of \( \kappa \) are preserved, so inside \( V[G_1][G_2][G_3] \), SDHL holds on a stationary subset of \( \kappa \). Thus, SDHL holds at \( \kappa \) in this model.

Question: does SDHL have any large cardinal strength (beyond that of a strongly inaccessible, which is needed for the definition)?


Thank You!