SACKS FORCING AND
THE SHRINK WRAPPING PROPERTY

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Abstract. We consider a property stronger than the Sacks property which holds between the ground model and the Sacks forcing extension.

1. The Shrink Wrapping Property

Suppose that \( V \) is a Sacks forcing extension of a model \( M \). Then the Sacks property holds between \( V \) and \( M \). That is, for each \( x \in \omega^{\omega} \), there exists a tree \( T \subseteq \omega^{\omega} \) in \( M \) such that \( x \in [T] \) and each level of \( T \) is finite. The following is a stronger property that we might want to hold between \( V \) and \( M \): for every sequence \( X = \langle x_n \in \omega^{\omega} : n < \omega \rangle \) there exists a sequence of trees \( \langle T_n \subseteq \omega^{\omega} : n < \omega \rangle \) such that

1) \((\forall n \in \omega) x_n \in [T_n] \);
2) \((\forall n_1, n_2 \in \omega) \) one of the following holds:
   a) \( x_{n_1} = x_{n_2} \);
   b) \( [T_{n_1}] \cap [T_{n_2}] = \emptyset \).

Unfortunately, if the sequence \( X \) is such that \( \langle (n_1, n_2) : x_{n_1} = x_{n_2} \rangle \notin M \), then there can be no such sequence of trees in \( M \). Thus, we need a weaker notion: a shrink wrapper.

In this next definition, we fix a canonical bijection \( \eta : \omega \to [\omega]^2 \) so that for each \( \tilde{n} \in \omega \), we may talk about the \( \tilde{n} \)-th pair \( \eta(\tilde{n}) \in [\omega]^2 \). The idea is that for each \( \{n_1, n_2\} = \eta(\tilde{n}) \in [\omega]^2 \), the functions \( F_{\tilde{n}, n_1} \) and \( F_{\tilde{n}, n_2} \), together with the finite sets \( I(n_1) \) and \( I(n_2) \), separate \( x_{n_1} \) and \( x_{n_2} \) as much as possible. For \( n \in \eta(\tilde{n}) \), the function \( F_{\tilde{n}, n} : \omega^2 \to \mathcal{P}(\omega^{\omega}) \) is shrink-wrapping \( 2^{\tilde{n}} \) possibilities for the value of \( x_n \). We need to make sure that what contains one possibility for \( x_{n_1} \) is sufficiently disjoint from what contains another possibility for \( x_{n_2} \), even if it is not possible that simultaneously both \( x_{n_1} \) and \( x_{n_2} \) are in the respective containers.

The main idea of shrink-wrapping is that for a fixed \( \tilde{n} \), if \( \{n_1, n_2\} \) is the \( \tilde{n} \)-th pair, the trees \( F_{\tilde{n}, n_1}(s) \) for \( s \in \omega^2 \) and \( F_{\tilde{n}, n_2}(s) \) for \( s \in \omega^2 \) separate \( x_{n_1} \) from \( x_{n_2} \) as much as possible. If \( x_{n_1} = x_{n_2} \), they
certainly cannot be separated and this is a special case. For the pair \( \{x_1, x_2\} \), there are finitely many “isolated” points which might prevent the separation of \( x_1 \) from \( x_2 \). In fact, we can get a finite set \( I(k) \) of isolated points associated to each \( x_k \) as opposed to each pair \( \{x_{n_1}, x_{n_2}\} \).

**Definition 1.1.** A shrink wrapper \( \mathcal{W} \) for \( \mathcal{X} = \langle x_n \in \omega : n \in \omega \rangle \) is a pair \( \langle \mathcal{F}, I \rangle \) such that \( I : \omega \to [\omega]^{<\omega} \) and \( \mathcal{F} \) is a collection of functions \( F_{\bar{n},n} \) for \( \bar{n} \in \omega \) and \( n \in \eta(\bar{n}) \) which satisfy the following conditions.

1. \( F_{\bar{n},n} : \bar{n}2 \to \mathcal{P}(\omega^\omega) \) and each element of \( \text{Im}(F_{\bar{n},n}) \) is a leafless subtree of \( \omega^\omega \) all of whose levels are finite.
2. \( (\exists s \in \bar{n}2) x_n \in [F_{\bar{n},n}(s)] \).
3. Given \( \{n_1, n_2\} = \eta(\bar{n}) \), \( (\forall s_1, s_2 \in \bar{n}2 \) one of the following relationships holds between the sets \( C_1 := [F_{\bar{n},n_1}(s_1)] \) and \( C_2 := [F_{\bar{n},n_2}(s_2)] \):
   a. \( C_1 = C_2 \) and if either \( x_{n_1} \in C_1 \) or \( x_{n_2} \in C_2 \), then \( x_{n_1} = x_{n_2} \);
   b. \( (\exists x \in I(n_1) \cap I(n_2)) C_1 = C_2 = \{x\} \);
   c. \( C_1 \cap C_2 = \emptyset \), and moreover there is an \( l \in \omega \) such that all elements of \( C_1 \) deviate from all elements of \( C_2 \) before level \( l \).

We will show that if \( V \) is a Sacks forcing extension of \( M \) and \( \mathcal{X} = \langle x_n \in \omega : n < \omega \rangle \) is in \( V \), there is a shrink wrapper \( \mathcal{W} \) for \( \mathcal{X} \) in \( M \). Also in this situation, we can build the shrink wrapper to satisfy the following additional property for all \( \bar{n} \in \omega \) and \( n \in \eta(\bar{n}) \):

4. \( (\forall s_1, s_2 \in \bar{n}2 \) one of the following relationships holds between the sets \( C_1 := [F_{\bar{n},n}(s_1)] \) and \( C_2 := [F_{\bar{n},n}(s_2)] \):
   a. \( (\exists x \in I(n)) C_1 = C_2 = \{x\} \);
   b. \( C_1 \cap C_2 = \emptyset \), and moreover there is an \( l \in \omega \) such that all elements of \( C_1 \) deviate from all elements of \( C_2 \) before level \( l \).

Note this is a requirement on the single function \( F_{\bar{n},n} \) where \( n \in \eta(\bar{n}) \), and not a requirement on the pair of functions \( \langle F_{\bar{n},n_1}, F_{\bar{n},n_2} \rangle \) where \( \{n_1, n_2\} = \eta(\bar{n}) \).

**Definition 1.2.** A forcing \( \mathbb{P} \) has the shrink wrapping property iff every sequence \( \mathcal{X} = \langle x_n \in \omega^\omega : n \in \omega \rangle \) in the forcing extension has a shrink wrapper \( \mathcal{W} \) in the ground model.

In Theorem 3.7 we will show that Sacks forcing has the shrink wrapping property. If a forcing has the shrink wrapping property, then it has the Sacks property (consider the sequence \( \langle x_n(0) \in \omega : n \in \omega \rangle \)). That is, consider a fixed real \( x \in \omega^\omega \). Fix any sequence \( \mathcal{X} = \langle x_n \in \omega^\omega : n \in \omega \rangle \) such that \( (\forall n \in \omega) x_n(0) = x(n) \). Let \( \mathcal{W} \) be a shrink
wrapper for $\mathcal{X}$. For each $n \in \omega$, let $\bar{n}_n \in \omega$ be the smallest number $\bar{n}$ such that $n$ is a member of the $\bar{n}$-th pair. Let $S_n$ be the set of all $i \in \omega$ such that the node $\langle i \rangle \in \leq^* \omega$ is on the 1-st level of set tree $F_{\bar{n}_n,n}(s)$ for some $s \in \bar{n}_n2$. We have that $S_n$ is a finite set of natural numbers and $x(n) = x_n(0) \in S_n$. Furthermore, the sequence $\langle S_n : n \in \omega \rangle$ is in any transitive model of ZF which contains the shrink wrapper $W$. This shows that the shrink wrapper property implies the Sacks property.

2. Application to Pointwise Eventual Domination

Before we show that there is always a shrink wrapper in the ground model after doing Sacks forcing, let us discuss an application of shrink wrappers themselves. Given two functions $f, g : \omega \omega \rightarrow \omega \omega$, let us write $f \leq^* g$ and say that $g$ pointwise eventually dominates $f$ iff $(\forall x \in \omega)(\forall \infty n) f(x)(n) \leq g(x)(n)$.

One may ask what is the cofinality of the set of Borel functions from $\omega \omega$ to $\omega \omega$ ordered by $\leq^*$. That answer is $2^{\omega}$, which follows from the result in [Hathaway] that given each $A \subseteq \omega$ there is a Baire class one (and therefore Borel) function $f_A : \omega \omega \rightarrow \omega \omega$ such that given any Borel $g : \omega \omega \rightarrow \omega \omega$ such that $f_A \leq^* g$, $A$ is $\Delta^1_1$ in any code for $g$. One may then ask what functions $f_A$ have this property.

Being precise, say that a function $f_A : \omega \omega \rightarrow \omega \omega$ sufficiently encodes $A \subseteq \omega$ if whenever $g : \omega \omega \rightarrow \omega \omega$ is Borel and satisfies $f_A \leq^* g$, then $A \in L[c]$ where $c$ is any code for $g$. What must a function do to sufficiently encode $A$? Given a sequence $\mathcal{X} = \langle x_n \in \leq^* \omega : n < \omega \rangle$, let us write $f_{\mathcal{X}} : \omega \omega \rightarrow \omega \omega$ for the function

$$f_{\mathcal{X}}(x)(n) := \begin{cases} \min\{l : x(l) \neq x_n(l)\} & \text{if } x \neq x_n, \\ 0 & \text{otherwise.} \end{cases}$$

Given $A \subseteq \omega$, is there some $\mathcal{X}$ such that $f_{\mathcal{X}}$ sufficiently encodes $A$? Using a shrink wrapper, we can show that the answer is no. Specifically, if $V$ is a Sacks forcing extension of $M$, $A \notin M$, and $\mathcal{X}$ is in $V$, then there is a shrink wrapper $W$ for $\mathcal{X}$ in $M$ which can be used to build a Borel function $g : \omega \omega \rightarrow \omega \omega$ (with a code in $M$) satisfying $f_{\mathcal{X}} \leq^* g$. Since $g$ has a code $c$ in $M$, $L[c] \subseteq M$ so $A \notin L[c]$.

To facilitate the discussion, let us make the following definition:

**Definition 2.1.** Given a tree $T \subseteq \leq^* \omega$, $\text{Exit}(T) : \omega \omega \rightarrow \omega$ is the function

$$\text{Exit}(T)(x) := \min\{l : x \in l \notin T\}.$$
In this section we will show that if $M$ is a transitive model of ZF and a sequence $\mathcal{X}$ of reals has a shrink wrapper in $M$, then there is a Borel function $g$ with a code in $M$ such that $f_\mathcal{X} \leq^* g$. We will warm up to this by first considering the situation when $M$ contains something stronger than a shrink wrapper. This will illustrate the main ideas.

**Proposition 2.2.** Let $\mathcal{X} = \langle x_n \in {}^\omega \omega : n \in \omega \rangle$. Suppose

$$T = \langle T_n : n \in \omega \rangle \in M$$

is a sequence of subtrees of $<\omega \omega$ satisfying the following:

1) $(\forall n \in \omega) x_n \in [T_n]$.
2) $(\forall n_1, n_2 \in \omega)$ one of the following holds:
   a) $x_{n_1} = x_{n_2}$;
   b) $[T_{n_1}] \cap [T_{n_2}] = \emptyset$.

Then there is a Borel function $g : \omega \omega \to \omega$ that has a Borel code in $M$ satisfying

$$(\forall x \in \omega \omega) f_\mathcal{X}(x) \leq^* g(x).$$

**Proof.** Let $g : \omega \omega \to \omega$ be defined by

$$g(x)(n) := \max\{\text{Exit}(T_n)(x), n\}.$$  

Certainly $g$ is Borel, with a code in $M$ (because $T \in M$). The “Exit($T_n$)(x)” part of the definition is doing most of the work. Specifically, for any $n \in \omega$ and $x \not\in [T_n]$,

$$f_\mathcal{X}(x)(n) = \text{Exit}([x_n])(x) \leq \text{Exit}(T_n)(x).$$

This is because since $x_n$ is a path through the tree $T_n$, $x \not\in [T_n]$ implies the level where $x$ exits $T_n$ is not before the level where $x$ differs from $x_n$. Thus, we have

$$(\forall n \in \omega) x \not\in [T_n] \Rightarrow f_\mathcal{X}(x)(n) \leq g(x)(n).$$

Suppose, towards a contradiction, that there is some $x \in {}^\omega \omega$ satisfying $f_\mathcal{X}(x) \not\leq^* g(x)$. Fix such an $x$. Let $A$ be the infinite set

$$A := \{n \in \omega : f_\mathcal{X}(x)(n) > g(x)(n)\}.$$  

It must be that $x \in [T_n]$ for each $n \in A$. By hypothesis, this implies $x_{n_1} = x_{n_2}$ for all $n_1, n_2 \in A$. Thus, $f_\mathcal{X}(x)(n)$ is the same constant for all $n \in A$. This is a contradiction, because $g(x)(n) \geq n$ for all $n$. \hfill $\square$

Here is the main result of this section:

**Theorem 2.3.** Let $\mathcal{X} = \langle x_n \in {}^\omega \omega : n \in \omega \rangle$. Suppose

$$\mathcal{W} = \langle \mathcal{F}, I \rangle \in M$$

...
is a shrink wrapper for $X$. Then there is a Borel function $g : \omega \omega \to \omega \omega$ that has a Borel code in $M$ satisfying

$$(\forall x \in \omega \omega) f_X(x) \leq^* g(x).$$

**Proof.** For each $n \in \omega$, let $T_n \subseteq \omega$ be the tree

$$T_n := \bigcap \{\bigcup \{\text{Im}(F_{\tilde{n},n}) : \tilde{n} \in \omega \land n \in \eta(\tilde{n})\}. $$

That is, for each $t \in \omega \omega$, $t \in T_n$ iff

$$(\forall \tilde{n} \in \omega)[n \in \eta(\tilde{n}) \Rightarrow t \in \bigcup_{s \in \tilde{n} 2} F_{\tilde{n},n}(s)]. $$

By part 2) of the definition of a shrink wrapper,

$$(\forall n \in \omega) x_n \in [T_n].$$

Let $e(n_2)$ be the least level $l$ such that if $n_1 < n_2$, $\tilde{n}$ satisfies $\eta(\tilde{n}) = \{n_1, n_2\}$, and $s_1, s_2 \in \tilde{n} \in \tilde{n} 2$ satisfy $[F_{\tilde{n},n_1}(s_1)] \cap [F_{\tilde{n},n_2}(s_2)] = \emptyset$, then all elements of $[F_{\tilde{n},n_1}(s_1)]$ deviate from all elements of $[F_{\tilde{n},n_2}(s_2)]$ before level $l$. Let $g : \omega \omega \to \omega \omega$ be defined by

$$g(x)(n) := \max\{\text{Exit}(T_n)(x), e(n), n\}. $$

Certainly $g$ is Borel, with a code in $M$ (because $\mathcal{W} \in M$). Just like in the previous proposition, since $x_n \in \omega \omega$, for all $x \in \omega \omega$ and $n \in \omega$ we have

$$x \not\in [T_n] \Rightarrow f_X(x)(n) \leq g(x)(n). $$

Suppose, towards a contradiction, that there is some $x \in \omega \omega$ satisfying $f_X(x) \not\leq^* g(x)$. Fix such an $x$. Let $A$ be the infinite set

$$A := \{n \in \omega : f_X(x)(n) > g(x)(n)\}. $$

It must be that $x \in [T_n]$ for each $n \in A$. Since $A$ is infinite, we may fix $n_1, n_2 \in A$ satisfying the following:

i) $n_1 < n_2$;

ii) $f_X(x)(n_1) \leq n_2$.

Let $\tilde{n}$ satisfy $\eta(\tilde{n}) = \{n_1, n_2\}$. Since $x \in [T_{n_1}]$, fix some $s_1' \in \tilde{n} 2$ satisfying

$$x \in [F_{\tilde{n},n_1}(s_1')] =: C_1. $$

Also, since $x_{n_2} \in [T_{n_2}]$, fix some $s_2 \in \tilde{n} 2$ satisfying

$$x_{n_2} \in [F_{\tilde{n},n_2}(s_2)] =: C_2. $$

By the definition of $e(n_2)$ and the fact that $\text{Exit}([x_{n_2}]) := e(n_2)$, it cannot be that $C_1 \cap C_2 = \emptyset$. Thus, by part 3) of the definition of a separation device, one of the following holds:
Now, b) cannot be the case because $C_2 = \{ x \}$ implies $x_{n_2} = x$, which implies $f_X(x)(n_2) = 0$, which contradicts the fact that $f_X(x)(n_2) > g(x)(n_2)$. On the other hand, a) cannot be the case because $x_{n_1} = x_{n_2}$ implies $f_X(x)(n_1) = f_X(x)(n_2)$, which by ii) implies $f_X(x)(n_2) = f_X(x)(n_2) \leq n_2 \leq g(x)(n_2) < f_X(x)(n_2)$, which is impossible. □

3. Sacks Forcing

In this section we will show that if $V$ is a Sacks forcing extension of a transitive model $M$ of ZF and $\mathcal{X} = \langle x_n : n \in \omega \rangle$ is an arbitrary sequence, then there is a shrink wrapper for $\mathcal{X}$ in $M$.

**Definition 3.1.** A tree $p \subseteq \omega^2$ is perfect if it is nonempty and for each $t \in p$, there are incomparable $t_1, t_2 \in p$ extending $t$. Sacks forcing $\mathbb{S}$ is the poset of all perfect trees $p \subseteq \omega^2$ where $p_1 \leq p_2$ iff $p_1 \subseteq p_2$.

Given $p_1, p_2 \in \mathbb{S}$, $p_1 \perp p_2$ means that $p_1$ and $p_2$ are incompatible.

**Definition 3.2.** Let $p \subseteq \omega^2$ be a perfect tree. A node $t \in p$ is called a branching node if $t \upharpoonright 0, t \upharpoonright 1 \in p$. Stem($p$) is the unique branching node $t$ of $p$ such that all elements of $p$ are comparable to $t$. A node $t \in p$ is said to be an $n$-th branching node if it is a branching node and there are exactly $n$ branching nodes that are proper initial segments of it. In particular, Stem($p$) is the unique 0-th branching node of $p$. Given Sacks conditions $p, q$, we write $q \leq_n p$ if $q \leq p$ and all of the $k$-th branching nodes, for $k \leq n$, of $p$ are in $q$ and are branching nodes.

**Lemma 3.3** (Fusion Lemma). Let $\langle p_n : n \in \omega \rangle$ be a sequence of Sacks conditions such that

$$p_0 \geq_0 p_1 \geq_1 p_2 \geq_2 \ldots.$$ 

Then $p_\omega := \bigcap_{n \in \omega} p_n$ is a Sacks condition below each $p_n$.

**Proof.** This is standard and can be found in introductory presentations of Sacks forcing. See, for example, [1]. □

The sequence $\langle p_n : n \in \omega \rangle$ in the above lemma is known as a fusion sequence. The following will help in the construction of fusion sequences.

**Lemma 3.4** (Fusion Helper Lemma). Let $\mathbb{S}$ be Sacks forcing. Let $R : \omega^2 \to \mathbb{S}$ be a function with the following properties:
1) $(\forall s_1, s_2 \in <\omega^2) s_2 \supseteq s_1$ implies $R(s_2) \leq R(s_1)$;
2) $(\forall s \in <\omega) \text{Stem}(R(s^0)) \perp \text{Stem}(R(s^\bot))$.

For each $n \in \omega$, let $p_n$ be the Sacks condition

$$p_n := \bigcup \{ R(s) : s \in n^2 \}.$$ 

Then

$$R(\emptyset) = p_0 \geq p_1 \geq p_2 \geq p_3 \geq 2 \ldots$$

is a fusion sequence.

**Proof.** Consider any $n \geq 1$. Certainly $p_n \supseteq p_{n+1}$, because for each $s \in n^2$, $R(s) \supseteq R(s^0) \cup R(s^\bot)$. To show that $p_n \geq p_{n+1}$, consider a $k$-th branching node $t$ of $p_n$ for some $k \leq n - 1$. One can check that there is some $s \in k^2$ such that $t$ is the largest common initial segment of $\text{Stem}(R(s^0))$ and $\text{Stem}(R(s^\bot))$. Since

$$\text{Stem}(R(s^0)) \cup \text{Stem}(R(s^\bot)) \subseteq R(s^0) \cup R(s^\bot) \subseteq p_{n+1},$$

we have that $t$ is a branching node of $p_{n+1}$. Thus, we have shown that for each $k \leq n - 1$, each $k$-th branching node of $p_n$ is a branching node of $p_{n+1}$. Hence, $p_n \geq p_{n+1}$.

We present a forcing lemma that is a basic building block for separating $x_{n_1}$ from $x_{n_2}$. Combining this with a fusion argument gives us the result.

**Lemma 3.5.** Let $\mathbb{P}$ be any forcing. Let $p_0, p_1 \in \mathbb{P}$ be conditions. Let $\check{\tau}_0, \check{\tau}_1$ be names for elements of $\omega^\omega$. Suppose that there is no $x \in \omega^\omega$ satisfying the following two statements:

1) $p_0 \forces \check{\tau}_0 = \check{x}$;
2) $p_1 \forces \check{\tau}_1 = \check{x}$.

Then there exist $p'_0 \leq p_0; p'_1 \leq p_1$ and $t_0, t_1 \in <\omega^\omega$ satisfying the following:

3) $t_0 \perp t_1$,
4) $p'_0 \forces \check{\tau}_0 \supseteq \check{t}_0$,
5) $p'_1 \forces \check{\tau}_1 \supseteq \check{t}_1$.

**Proof.** There are two cases to consider. The first is that there exists some $x \in \omega^\omega$ such that 1) is true. When this happens, 2) is false. Hence, there exist $t_1 \in <\omega^\omega$ and $p'_1 \leq p_1$ such that 5) is true and $x \perp t_1$. Letting $p'_0 := p_0$ and $t_0$ be some initial segment of $x$ incompatible with $t_1$, we see that 3) and 4) are true.

The second case is that there is no $x \in \omega^\omega$ satisfying 1). When this happens, there exist conditions $p'_0, p'_1 \leq p_0$ and incompatible nodes $s_a, s_b \in <\omega^\omega$ satisfying both $p'_0 \forces \check{\tau}_0 \supseteq \check{s}_a$ and $p'_0 \forces \check{\tau}_0 \supseteq \check{s}_b$. Now, it
cannot be that both $p_1 \Vdash \tilde{\tau}_1 \supseteq \delta_a$ and $p_1 \Vdash \tilde{\tau}_1 \supseteq \delta_b$. Assume, without loss of generality, that $p_1 \not\Vdash \tilde{\tau}_1 \supseteq \delta_a$. This implies that there exist $p'_1 \leq p_1$ and $t_1 \in ^\omega \omega$ such that $s_a \perp t_1$ and $p'_1 \Vdash \tilde{\tau}_1 \supseteq \vec{t}_1$. Letting $p'_0 := p'_0$ and $t_0 := s_a$, we are done. \qedhere

At this point, the reader may want to think about how to use this lemma to prove that if $V$ is a Sacks forcing extension of $M$ and $X = \langle x_n : n \in \omega \rangle$ satisfies

$$\forall n \in \omega \ x_n \notin M$$

and

$$\{\langle n_1, n_2 \rangle : x_{n_1} = x_{n_2} \} \in M,$$

then there is a sequence $T$ of subtrees of $^\omega \omega$ satisfying the hypotheses of Proposition 2.2. The next lemma explains the appearance of $I$ in the definition of a separation device. We are intending the name $\check{\tau}$ to refer to the $x_n$ in the sequence $X = \langle x_n : n \in \omega \rangle$.

**Lemma 3.6.** Consider Sacks forcing $S$. Let $p \in S$ be a condition and $\check{\tau}$ a name satisfying $p \Vdash \check{\tau} : \omega \rightarrow ^\omega \omega$. Then there exists a condition $p' \leq p$ and there exists a function $I : \omega \rightarrow [^\omega \omega]^{<\omega}$ satisfying

$$p' \Vdash \forall n \in \omega \check{\tau}(n) \in \check{V} \rightarrow \check{\tau}(n) \in \check{I}(n).$$

**Proof.** We may easily construct a function $R : \omega \rightarrow S$ that satisfies the conditions of Lemma 3.4 such that $R(\emptyset) \leq p$ and for each $s \in n^2$, either $R(s) \Vdash \check{\tau}(n) \notin \check{V}$ or $(\exists x \in ^\omega \omega) R(s) \Vdash \check{\tau}(n) = \check{x}$. Define $I$ as follows:

$$I(n) := \{x \in ^\omega \omega : (\exists s \in n^2) R(s) \Vdash \check{\tau}(n) = \check{x}\}.$$

Let $p' := \bigcap_n \bigcup \{R(s) : s \in n^2\}$. The condition $p'$ and the function $I$ are as desired. \qedhere

We are now ready for the main forcing argument of this section.

**Theorem 3.7.** Consider Sacks forcing $S$. Let $p \in S$ be a condition and $\check{\tau}$ be a name satisfying $p \Vdash \check{\tau} : \omega \rightarrow ^\omega \omega$. Then there exists a condition $q \leq p$ and there exists a pair $\mathcal{W} = \langle \mathcal{F}, I \rangle$ satisfying

$q \Vdash \check{\mathcal{W}}$ is a shrink wrapper for $\langle \check{\tau}(n) : n \in \omega \rangle$.

**Proof.** First, let $p' \leq p$ and $I : \omega \rightarrow [^\omega \omega]^{<\omega}$ be given by the lemma above. That is, for each $n \in \omega$,

$$p' \Vdash \check{\tau}(\check{n}) \in \check{V} \rightarrow \check{\tau}(\check{n}) \in \check{I}(\check{n}).$$

We will define a function $R : [^\omega 2 \rightarrow S$ with $R(\emptyset) \leq p'$ satisfying conditions 1) and 2) of Lemma 3.4. At the same time, we will construct
a family of functions
\[ \mathcal{F} = \{ F_{\tilde{n}, n} : \tilde{n} \in \omega, n \in \eta(\tilde{n}) \}. \]

Our \( q \) will be
\[ q := \bigcap_{\tilde{n}} \bigcup_{s \in ^{\tilde{n}} \omega} R(s). \]

The function \( F_{\tilde{n}, n} \) will return a leafless subtree of \( ^{<\omega} \omega \). We will have it so for all \( n \in \omega \) and all \( \tilde{n} \) satisfying \( n \in \eta(\tilde{n}) \),
\[ (\forall s \in ^{\tilde{n}} \omega) R(s) \models \dot{\tau}(\tilde{n}) \in [\tilde{F}_{\tilde{n}, n}(\dot{s})]. \]

Thus, \( q \) will easily force that \( \mathcal{W} \) satisfies conditions 1) and 2) of the definition of a shrink wrapper. To show that \( q \) forces condition 3) of that definition, it suffices to show that for all \( \{ n_1, n_2 \} = \eta(\tilde{n}) \) and all \( s_1, s_2 \in ^{\tilde{n}} \omega \), one of the following holds, where \( T_1 := F_{\tilde{n}, n_1}(s_1) \) and \( T_2 := F_{\tilde{n}, n_2}(s_2) \):

1. \( T_1 = T_2 \) and \( (\forall s \in ^{\tilde{n}} \omega) \),
\[ R(s) \models (\dot{\tau}(\tilde{n}_1) \in [\tilde{T}_1] \lor \dot{\tau}(\tilde{n}_2) \in [\tilde{T}_2]) \rightarrow \dot{\tau}(\tilde{n}_1) = \dot{\tau}(\tilde{n}_2); \]

2. \( (\exists x \in I(n_1) \cap I(n_2)) [T_1] = [T_2] = \{ x \}; \)

3. \( [T_1] \cap [T_2] = \emptyset \), and moreover Stem\( (T_1) \subseteq \text{Stem}(T_2) \).

We will define the functions \( F_{\tilde{n}, n} \) and the conditions \( R(s) \) for \( s \in ^{\tilde{n}} \omega \) by induction on \( \tilde{n} \). Beginning at \( \tilde{n} = 0 \), let \( \{ n_1, n_2 \} = \eta(0) \). We will define \( F_{0, n_1} : ^{0} \omega \rightarrow \mathcal{S} \), \( F_{0, n_2} : ^{0} \omega \rightarrow \mathcal{S} \), and \( R(0) \leq p' \). If \( p' \models \dot{\tau}(\tilde{n}_1) = \dot{\tau}(\tilde{n}_2) \), then let \( R(0) := p' \) and define \( F_{0, n_1}(0) = F_{0, n_2}(0) = T \) where \( T \subseteq ^{<\omega} \omega \) is any leafless tree satisfying \( p' \models \dot{\tau}(\tilde{n}_1) \in [\tilde{T}] \). This causes 3a) to be satisfied. If \( p' \not\models \dot{\tau}(\tilde{n}_1) = \dot{\tau}(\tilde{n}_2) \), then let \( t_1, t_2 \in ^{<\omega} \omega \) be incomparable nodes and let \( R(0) \leq p' \) satisfy \( R(0) \models \dot{\tau}(\tilde{n}_1) \subseteq t_1 \) and \( R(0) \models \dot{\tau}(\tilde{n}_2) \subseteq t_2 \). Then we may define \( F_{0, n_1}(0) = T_1 \) and \( F_{0, n_2}(0) = T_2 \) where \( T_1 \) and \( T_2 \) are leafless trees such that Stem\( (T_1) \subseteq t_1, \text{Stem}(T_2) \subseteq t_2, R(0) \models \dot{\tau}(\tilde{n}_1) \in [\tilde{T}_1], \text{and } R(0) \models \dot{\tau}(\tilde{n}_2) \in [\tilde{T}_2] \). This causes 3c') to be satisfied.

We will now handle the successor step of the induction. Let \( \{ n_1, n_2 \} = \eta(\tilde{n}) \) for some \( \tilde{n} > 0 \). We will define \( R(s) \) for each \( s \in ^{\tilde{n}} \omega \), and we will define both \( F_{\tilde{n}, n_1} \) and \( F_{\tilde{n}, n_2} \) assuming \( R(s') \) has been defined for each \( s' \in ^{\tilde{n}} \omega \). To keep the construction readable, we will start with initial values for the \( R(s') \)’s and the \( F_{\tilde{n}, n} \)’s, and we will modify them as the construction progresses until we arrive at their final values. That is, we will say “replace \( R(s) \) with a stronger condition...” and “shrink the tree \( F_{\tilde{n}, n}(s) \)...”. When we make these replacements, it is understood that still \( R(s) \models \dot{\tau}(\tilde{n}) \in [\tilde{F}_{\tilde{n}, n}(\dot{s})] \). The construction consists of 5 steps.
Step 1: First, for each \( s \in (\bar{n} - 1)/2 \), let \( R(s \sim 0) \) and \( R(s \sim 1) \) be arbitrary extensions of \( R(s) \) such that \( \text{Stem}(R(s \sim 0)) \perp \text{Stem}(R(s \sim 1)) \). Also, for each \( n \in \{n_1, n_2\} \) and \( s \in \bar{n}/2 \), let \( F_{\bar{n}, n}(s) \) be an arbitrary leafless subtree of \( \ll \omega \omega \) such that \( R(s) \models \bar{\tau}(\bar{n}) \in [F_{\bar{n}, n}(s)] \).

Step 2: For each \( s \in \bar{n}/2 \) and \( n \in \{n_1, n_2\} \), strengthen \( R(s) \) so that either \( \bar{R}(s) \models \bar{\tau}(\bar{n}) \notin \bar{V} \) or \( (\exists x \in I(n)) R(s) \models \bar{\tau}(\bar{n}) = \check{x} \). If the latter case holds, shrink \( F_{\bar{n}, n}(s) \) so that it has only one path.

Step 3: Fix \( n \in \{n_1, n_2\} \). For each pair of distinct \( s_1, s_2 \in \bar{n}/2 \), strengthen each \( R(s_1) \) and \( R(s_2) \) and shrink each \( F_{\bar{n}, n}(s_1) \) and \( F_{\bar{n}, n}(s_2) \) so that one of the following holds:

i) \( (\exists x \in I(n)) [F_{\bar{n}, n}(s_1)] = [F_{\bar{n}, n}(s_2)] = \{x\} \);

ii) \( \text{Stem}(F_{\bar{n}, n}(s_1)) \perp \text{Stem}(F_{\bar{n}, n}(s_2)) \).

That is, if i) cannot be satisfied, then we may use Lemma 3.5 to satisfy ii).

Step 4: For each pair of distinct \( s_1, s_2 \in \bar{n}/2 \) such that either \( R(s_1) \models \bar{\tau}(\bar{n}_1) \notin \bar{V} \) or \( R(s_2) \models \bar{\tau}(\bar{n}_2) \notin \bar{V} \), use Lemma 3.5 to strengthen \( R(s_1) \) and \( R(s_2) \) and shrink \( F_{\bar{n}, n_1}(s_1) \) and \( F_{\bar{n}, n_1}(s_1) \) so that

\[
\text{Stem}(F_{\bar{n}, n_1}(s_1)) \perp \text{Stem}(F_{\bar{n}, n_2}(s_2)).
\]

Step 5: For each \( s \in \bar{n}/2 \), do the following: If \( R(s) \models \bar{\tau}(\bar{n}_1) = \bar{\tau}(\bar{n}_2) \), then replace both \( F_{\bar{n}, n_1}(s) \) and \( F_{\bar{n}, n_2}(s) \) with \( F_{\bar{n}, n_1}(s) \cap F_{\bar{n}, n_2}(s) \). Otherwise, strengthen \( R(s) \) and shrink \( F_{\bar{n}, n_1}(s) \) and \( F_{\bar{n}, n_2}(s) \) so that

\[
\text{Stem}(F_{\bar{n}, n_1}(s)) \perp \text{Stem}(F_{\bar{n}, n_2}(s)).
\]

This completes the construction of \( \{R(s) : s \in \bar{n}/2\} \), \( F_{\bar{n}, n_1} \), and \( F_{\bar{n}, n_2} \).

We will now prove that it works. Fix \( s_1, s_2 \in \bar{n}/2 \) and let \( T_1 := F_{\bar{n}, n_1}(s_1) \) and \( T_2 := F_{\bar{n}, n_2}(s_2) \). We must show that one of 3a', 3b', or 3c' holds. The cleanest way to do this is to break into cases depending on whether \( s_1 = s_2 \) or not.

Case \( s_1 \neq s_2 \): If either \( R(s_1) \models \bar{\tau}(\bar{n}_1) \notin \bar{V} \) or \( R(s_2) \models \bar{\tau}(\bar{n}_2) \notin \bar{V} \), then by Step 4, we see that 3c' holds. Otherwise, by Step 2, \( (\exists x \in I(n_1)) [T_1] = \{x\} \) and \( (\exists x \in I(n_1)) [T_2] = \{x\} \). Hence, easily either 3b' or 3c' holds.

Case \( s_1 = s_2 \): If \( R(s_1) \not\models \bar{\tau}(\bar{n}_1) = \bar{\tau}(\bar{n}_2) \), then by Step 5, we see that 3c' holds. Otherwise, we are in the case that

\[
R(s_1) \models \bar{\tau}(\bar{n}_1) = \bar{\tau}(\bar{n}_2).
\]

By Step 5, \( T_1 = T_2 \). Now, if \( R(s_1) \models \bar{\tau}(\bar{n}_1) \in \bar{V} \), then of course also \( R(s_1) \models \bar{\tau}(\bar{n}_2) \in \bar{V} \), and by Step 2 we see that 3b' holds. Otherwise, \( R(s_1) \models \bar{\tau}(\bar{n}_1) \notin \bar{V} \). Hence, \( [T_1] \) is not a singleton. We will show that
3a') holds. Consider any \( s \in \tilde{n}_2 \). We must show
\[
R(s) \forces (\hat{\tau}(\tilde{n}_1) \in [\tilde{T}_1] \lor \hat{\tau}(\tilde{n}_2) \in [\tilde{T}_1]) \rightarrow \hat{\tau}(\tilde{n}_1) = \hat{\tau}(\tilde{n}_2).
\]
If \( s = s_1 \), we are done. Now suppose \( s \neq s_1 \). It suffices to show
\[
R(s) \forces \neg(\hat{\tau}(\tilde{n}_1) \in [\tilde{T}_1] \lor \hat{\tau}(\tilde{n}_2) \in [\tilde{T}_1]).
\]
That is, it suffices to show \( R(s) \not\forces \hat{\tau}(\tilde{n}_1) \in [\tilde{T}_1] \) and \( R(s) \not\forces \hat{\tau}(\tilde{n}_2) \in [\tilde{T}_1] \).

Since \( s \neq s_1 \) and \( [T_1] \) is not a singleton, by Step 3, \( \text{Stem}(F_{\tilde{n},n}(s)) \perp \text{Stem}(T_1) \). Recall that
\[
R(s) \forces \hat{\tau}(\tilde{n}_1) \in [\tilde{F}_{\tilde{n},n}(\tilde{s})].
\]
Hence, since \( [\tilde{F}_{\tilde{n},n}(\tilde{s})] \cap [T_1] = \emptyset \), \( R(s) \not\forces \hat{\tau}(\tilde{n}_1) \notin [\tilde{T}_1] \). By a similar argument, \( R(s) \not\forces \hat{\tau}(\tilde{n}_2) \notin [\tilde{T}_1] \). This completes the proof. \( \square \)

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