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Eigenvalue Solution of Thermoelastic Damping in Beam Resonators Using a Finite Element Analysis

A finite element formulation is developed for solving the problem related to thermoelastic damping in beam resonator systems. The perturbation analysis on the governing equations of heat conduction, thermoleasticity, and dynamic motion leads to a linear eigenvalue equation for the exponential growth rate of temperature, displacement, and velocity. The numerical solutions for a simply supported beam have been obtained and shown in agreement with the analytical solutions found in the literature. Parametric studies on a variety of geometrical and material properties demonstrate their effects on the frequency and the quality factor of resonance. The finite element formulation presented in this work has advantages over the existing analytical approaches in that the method can be easily extended to general geometries without extensive computations associated with the numerical iterations and the analytical expressions of the solution under various boundary conditions. [DOI: 10.1115/1.2748472]

Keywords: thermoelastic damping, beam resonator, eigenvalue, finite element method

1 Introduction

Thermoelastic damping is a phenomenon related to the irreversible heat dissipation induced by the coupling between heat transfer and strain rate during the compression and decompression of an oscillating system. In the past few years interest in both experimental and theoretical investigations of thermoelastic damping increased rapidly, especially in the research arena related to microelectromechanical systems (MEMS) and nanoelectromechanical systems (NEMS) [1–3]. This is mainly prompted by the pursuit of low-energy dissipation or high-quality factor in designing and fabricating high-precision actuators, sensors, and mechanical filters [4].

The earliest work related to thermoelastic damping can be traced back to Zener [5] who established a general theory of thermoelastic damping and derived an equation to relate the energy dissipation, or "internal friction" Q^{-1} (fraction of energy loss per radian of vibration) to the material properties and geometric parameters of a thin beam under bending. However, Zener's derivation involved some mathematical and physical simplifications. For example, the boundary conditions were expressed in a trigonometric series and a truncation was made on the series to the first term only. In addition, an assumption was involved in the derivation based on the fact that the "relaxation strength" (the difference between the adiabatic and isothermal values of the equivalent elastic modulus) is much less than 1 for a thin beam.

Lifshitz and Roukes [4], however, indicated that these simplifications are actually not necessary. They derived an exact expression of the solution for the same problem from the equations of motion and the fundamental theories of thermoelasticity, despite the fact that in most situations Zener's solution is an excellent approximation to the exact solution.

Although Lifshitz and Roukes' solution was quite successful in predicting thermoelastic damping of beam systems, it is questionable to apply the same solution to other geometries such as clamped plates, which have wide applications in micropumps and pressure sensors, and whose mode shapes vary across the plate width. Nayfeh and Younis [6] extended the technique to these cases and derived an analytical solution for the problem involving microplates of finite width.

Zener's classical work and Lifshitz and Roukes' solution technique were also successfully extended to solve the problem involving in-plane vibration of thin rings [7]. These rings are commonly used in fabrication of rate sensors (gyroscopes) [8].

In addition to the investigations of geometric effects, some researchers also indicated the importance of the boundary conditions. For example, Sun et al. [9] studied thermoelastic damping in beam resonators subjected to various boundary conditions, by combining the finite sine Fourier transformation method and the Laplace transformation method in the normal mode analysis.

All the above research activities were based on analytical approaches using the continuum theories. The advantage of the analytical approaches is obvious in that the results can be expressed either in the explicit form of an exact formula or as a set of nonlinear equations implicitly. In the latter case, the exact results can still be found via numerical iterations. However the disadvantages of the analytical approaches are also obvious: they can only deal with simple geometries such as beams and plates under uniform boundary conditions. The rapid advancements in MEMS technologies require designs and fabrications of components involving more and more complicated geometries subjected to inhomogeneous boundary conditions, with which an analytical approach will be impractical and a finite element discretization scheme will be a natural tool to overcome the associated difficulties.

Silver and Peterson [10] recently proposed using the finite element method to solve the thermoelastic damping problem for beams. They formulated the problem in a fashion similar to the finite element treatment on linearly viscoelastic structures originally suggested by Segalman [11]. In particular, the elastic stiffness matrix was augmented by a differential stiffness matrix, which was evaluated by a Fourier transform. This was among the first efforts applying the discretization method to solve a thermoelastic damping problem (although some earlier works exist in

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the literature [12]). However, the determination of the differential stiffness matrix requires the evaluation of Fourier transform in an integral form. In addition, although in principle one can extend this method to deal with geometries other than beams and trusses, it is unclear whether the method will be efficient and what kind of numerical accuracy it can achieve.

In the current work, a different approach is taken to solve the thermoelastic damping problem using a finite element formulation. In this approach, a perturbation of the temperature field is explicitly applied to the governing equations associated with heat transfer, thermoelasticity, and structural vibration. This results in an eigenvalue equation with the imaginary part of the eigenvalue representing the frequency of the harmonic vibration and the real part representing the decaying rate of the amplitude. The advantage of the method lies in that it is very easy to extend the same formulation to solve a problem with arbitrary geometry and complicated boundary conditions, although only those results related to the two-dimensional formulation are presented in the current work.

The method is actually analogous to the one recently used in the formulation of thermoelastodynamic instability (TEDI) problems involving frictional heating, in which the frictionally excited thermoelastic instability is coupled with dynamic vibration [13,14]. For the thermoelastic damping problem, once the problem is linearized (basically assuming that the temperature variations are small compared with the mean absolute temperature), it would be reduced to an eigenvalue problem very similar to TEDI, except that there is a thermoelastic coupling term in the heat conduction equation. The implementation of the finite element formulation on the thermoelastic damping problem is therefore a natural extension of the method developed for TEDI with some necessary modification.

2 Method

To demonstrate the proposed methodology, a finite element formulation of the problem in a general two-dimensional geometry is developed based on the fundamental theories of thermoelasticity and structural dynamics. Then the method is applied to the simplified case of a thin beam model so that the numerical results can be compared to the existing analytical solutions available in the literature.

2.1 The Heat Transfer Problem. The two-dimensional heat diffusion equation involving thermoelastic damping can be written as the following differential equation

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\rho C_p\frac{\partial T}{\partial t}+\frac{E\alpha T}{1-2\nu}\frac{\partial \bar{e}}{\partial t} \tag{1}$$

where k, ρ and c, E, α, and ν are thermal conductivity, density, specific heat of mass, elastic modulus, thermal expansion coefficient, and Poisson's ratio, respectively; $\bar{e}$ is the dilatation strain due to the thermal effect

$$\bar{e}=\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz} \tag{2}$$

where ε_{xx}, ε_{yy}, ε_{zz} are the strains in the x, y, and z directions. We assume an exponentially growing perturbation solution of the form

$$T(x,y,t)=T_0(x,y)+\mathcal{R}\{e^{bt}\Theta(x)\} \tag{3}$$

where b is a complex exponential growth rate; T_0 is the steady-state solution and satisfies the heat diffusion equation; and $\mathcal{R}$ represents the real part of a complex number. Applying the standard Galerkin finite element formulation results in a matrix equation in the following form

$$\mathbf{K\Theta}+b\mathbf{H\Theta}+b\mathbf{NFU}=0 \tag{4}$$

where the elemental matrices can be expressed as

$$\mathbf{K_e}=\int_\Omega k\left(\frac{\partial \mathbf{N}}{\partial x}\frac{\partial \mathbf{N}^T}{\partial x}+\frac{\partial \mathbf{N}}{\partial y}\frac{\partial \mathbf{N}^T}{\partial y}\right)dx\,dy \tag{5}$$

$$\mathbf{H_e}=\int_\Omega \rho C_p\mathbf{NN}^T dx\,dy \tag{6}$$

$$\mathbf{F}=[\mathbf{F_1F_2F_3F_4}] \tag{7}$$

where

$$\mathbf{F}_i=\frac{E\alpha T_0}{1-2\nu}\left(\frac{dN_i}{dx}+\frac{dN_i}{dy}\right)[1\ 1] \tag{8}$$

In the above integrals $\mathbf{N}(x,y)$ is the shape function and $\mathbf{N}_i$ ($i=1,2,3,4$) represents the four terms in the shape function $\mathbf{N}(x,y)$. (The finite elements used here are two-dimensional quadrilateral elements.) Note that the problem is linearized assuming small perturbation such that the temperature T in the thermoelastic coupling term has been replaced by the mean absolute temperature T_0.

2.2 Equation of Motion. The equations of motion for a two-dimensional elastic body are as follows

$$\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \sigma_{xy}}{\partial y}=\rho\frac{\partial^2 u_x}{\partial t^2} \tag{9}$$

$$\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \sigma_{xy}}{\partial x}=\rho\frac{\partial^2 u_y}{\partial t^2} \tag{10}$$

where σ_{xx}, σ_{xy}, and σ_{yy} represent the stresses; and u_x and u_y are the displacements.

2.3 Thermoelasticity. For a two-dimensional problem, either the plane-strain or plane-stress assumptions can be made. For simplicity, let us consider the plane-strain assumption only. The plane-stress problem can be formulated in a similar fashion. The constitutive law of thermoelasticity for an isotropic material is

$$\{\sigma_{xx}\ \sigma_{yy}\ \sigma_{xy}\}^T=\mathbf{C}\{\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{xy}\}^T-\mathbf{D}T \tag{11}$$

where

$$\mathbf{C}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1/2-\nu\end{bmatrix} \tag{12}$$

and

$$\mathbf{D}=\frac{E\alpha}{(1-2\nu)}\begin{bmatrix}1\\1\\0\end{bmatrix} \tag{13}$$

$\{\varepsilon\}$ is the strain vector, which can be expressed in the matrix form in terms of the nodal displacement vector $\mathbf{U}$. The result is

$$\{\varepsilon_{xx}\ \varepsilon_{yy}\ \varepsilon_{xy}\}^T=\mathbf{BU} \tag{14}$$

and

$$\mathbf{B}=[\mathbf{B_1\ B_2\ B_3\ B_4}] \tag{15}$$

in which

$$\mathbf{B}_i=\begin{bmatrix}\frac{dN_i}{dx} & 0\\ 0 & \frac{dN_i}{dy}\\ \frac{dN_i}{dy} & \frac{dN_i}{dx}\end{bmatrix},\quad i=1,2,3,4 \tag{16}$$

We assume the displacement in the perturbation form as

$$u(x,y,t) = u_0(x,y) + \mathcal{R}\{e^{bt}U(x,y)\} \tag{17}$$

and velocity as

$$u'(x,y,t) = u_0'(x,y) + \mathcal{R}\{e^{bt}U'(x,y)\} \tag{18}$$

These components have been assumed in the above expressions for the purpose of reducing the otherwise second-order eigenvalue problem (which is difficult to solve) to a first-order problem. (A second-order eigenvalue problem would be unavoidable if the velocity was not expressed in a perturbed form.)

Applying Eqs. (9)–(18), the equations of motion can be reduced to a matrix equation

$$\mathbf{LU} - \mathbf{G\Theta} + b\mathbf{MU}' = \mathbf{0} \tag{19}$$

where the elemental matrices are obtained as

$$\mathbf{L}_e = \int_\Omega \mathbf{B}^T\mathbf{CB}\, dx$$

$$\mathbf{G}_e = \int_\Omega \mathbf{B}^T\mathbf{DN}\, dx\, dy$$

$$\mathbf{M_e} = \int_\Omega \rho\mathbf{NN}^T dx\, dy \tag{20}$$

2.4 Eigenvalue Equation. Note that we have a relation between the displacement and velocity perturbations as

$$\mathbf{U}' = b\mathbf{U} \tag{21}$$

Combining Eqs. (4), (19), and (21) yields

$$\left(\begin{bmatrix} \mathbf{K} & 0 & 0 \\ \mathbf{G} & -\mathbf{L} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix} - b\begin{bmatrix} \mathbf{H} & \mathbf{NF} & 0 \\ 0 & 0 & \mathbf{M} \\ 0 & \mathbf{I} & 0 \end{bmatrix}\right)\begin{bmatrix} \mathbf{\Theta} \\ \mathbf{U} \\ \mathbf{U}' \end{bmatrix} = 0 \tag{22}$$

or

$$\tilde{\mathbf{A}}\mathbf{X} = \mathbf{b}\tilde{\mathbf{B}}\mathbf{X} \tag{23}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{K} & 0 & 0 \\ \mathbf{G} & -\mathbf{L} & 0 \\ 0 & 0 & \mathbf{I} \end{bmatrix};\ \tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{H} & \mathbf{NF} & 0 \\ 0 & 0 & \mathbf{M} \\ 0 & \mathbf{I} & 0 \end{bmatrix};\ \mathbf{X} = [\mathbf{\Theta},\mathbf{U},\mathbf{U}']^T \tag{24}$$

and $\mathbf{I}$ is an identity matrix. This is a generalized eigenvalue equation. The eigenvalue of the equation is the growth rate b and the eigenvector is $[\mathbf{\Theta},\mathbf{U},\mathbf{U}']^T$, i.e., nodal temperature, displacement, and velocity.

The effect of thermoelastic damping on the attenuation of the vibration can be expressed in the quality factor Q defined as

$$Q = \frac{1}{2}\left|\frac{\mathcal{I}(b)}{\mathcal{R}(b)}\right| \tag{25}$$

where $\mathcal{R}(b)$ and $\mathcal{I}(b)$ represent the real part and the imaginary part of b, respectively.

3 Numerical Results

3.1 Existing Analytical Solutions. Zener's solution [5] for thermoelastic damping in a thin beam can be approximated as

$$Q^{-1} = \Delta_E\frac{\omega_0\tau}{1 + (\omega_0\tau)^2} \tag{26}$$

where

Table 1 Beam properties of the standard model used in the analysis

Young's modulus, E (Pa)	2.0×10^{11}
Poisson's ratio, ν	0
Thermal expansion coefficient, α (K^{-1})	1.2×10^{-5}
Thermal conductivity, k (W/m·K)	42.0
Specific heat, C_p (J/kg·K)	2.0×10^3
Density, ρ (kg/m^3)	7800
Temperature, T (K)	300
Thickness, h (m)	0.05
Length, L (m)	2.0

$$\Delta_E = \frac{E\alpha^2T_0}{\rho C_p} \tag{27}$$

τ is the relaxation time determined by

$$\tau = \frac{h^2\rho C_p}{\pi^2 k} \tag{28}$$

and ω_0 is the undamped natural frequency for the relevant mode. For a simply supported beam, the first natural frequency is

$$\omega_0^2 = \frac{\pi^4Eh^2}{12\rho L^4} \tag{29}$$

Lifshitz and Roukes' precise solution [4] can be expressed in the following form

$$Q^{-1} = \frac{E\alpha^2T_0}{\rho C_p}\left(\frac{6}{\xi^2} - \frac{6}{\xi^3}\frac{\sin\xi + \sinh\xi}{\cos\xi + \cosh\xi}\right) \tag{30}$$

where

$$\xi = h\sqrt{\frac{\omega_0\rho C_p}{2k}} \tag{31}$$

3.2 Simple Convergence Tests. To validate the method proposed in this work, finite element analysis was performed on a simply supported thin beam to compute its eigenfrequency and the associated quality factor of resonance.

The analysis was performed using the two-dimensional quadrilateral plane-strain elements as formulated aforementioned. The geometric parameters (as shown in Table 1) were obtained from the benchmark eigenvalue problem used by the commercial software ANSYS. Although MEMS devices have much smaller sizes and the thermoelastic damping is size dependent (it can be shown that the quality factor is a function of $h^{3/2}L^{-1}$), the conclusions obtained from this geometry can apply to other problems at different size scales as well, as long as the results are appropriately normalized. The aspect ratio of the beam geometry (thickness divided by length) in the current model is 1:40 and Poisson's ratio of the material is set to zero for approximating the thin beam. The length was discretized into 200 elements in the standard model and ten elements were used in the thickness direction for reducing the mesh size. This is reasonable because of the small aspect ratio of the beam geometry. Increasing the element numbers beyond these values has proved difficult due to the limitation of the available computer facilities. The first natural frequency of the beam in the simplified case without damping (this can easily be achieved by setting $T_0=0$) was compared to Eq. (29), showing good agreement (Fig. 1). When the thermoelastic damping is present ($T_0=300$ K), deviations of the results from Eq. (29) for large thicknesses have been observed.

A convergence test was also performed to study the effects of element number on both quality factor and frequency, as shown in Fig. 2. It can be seen that there is a significant error in the result when the element number is less than 50 (the element length is larger than the beam thickness in this case). However, when the element number is greater than 100, little changes have been ob-

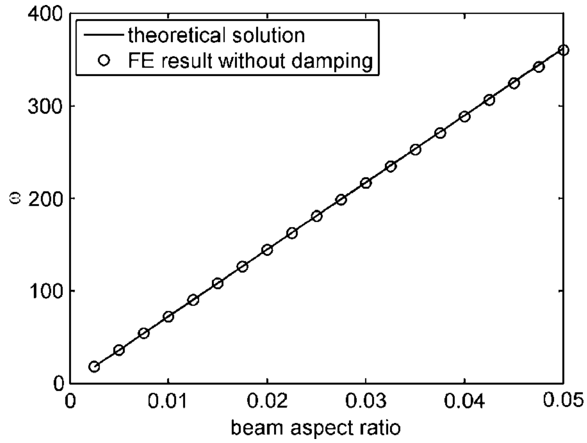

Fig. 1 Comparison of the finite element solution and exact solution for frequency of a simply supported thin beam. The theoretical solution corresponds to the undamped natural frequency.

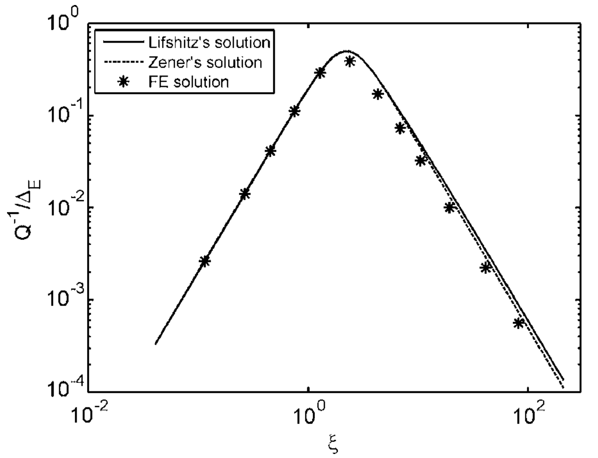

Fig. 3 Comparison of the normalized quality factor with the existing analytical solutions

served and the results are almost the same as the asymptotic solution. In fact, the numerical error associated with an insufficient element number can be easily explained by shear locking of the elements. It is known that bilinear quadrilaterals have a very poor bending performance and are prone to shear locking. This is only avoided if the length of a single element becomes much smaller than the beam thickness, which is confirmed by this computation.

3.3 Comparison Between the Numerical and Analytical Models. The quality factor was also compared with the existing analytical results from Zener's approximation and Lifshitz's solution for thin beams.

The parameters involved were normalized based on the same terminologies used by Lifshitz et al. [4] The variability in the parameters was achieved by varying the beam thickness. The result is presented in Fig. 3. The finite element solutions are in agreement with the analytical solutions, especially for small thicknesses. They also predict the identical peak value of the solution curve, which has validated the finite element formulation presented above. The differences at large ξ are induced by two sources: (1) The discretization scheme involved in the finite element method inevitably results in numerical inaccuracies. Due to the consideration of minimizing the computational effort, only 200 elements were used in the length direction and only ten elements were used in the thickness direction. The mesh might not be sufficiently fine for accurate computation at large ξ. (2) The theory of elasticity differs from the beam theory in that it considers the shear variation in the thickness direction as well (although the thickness is very small relative to the length in the present problem). Again, the numerical error associated with an insufficient element number as stated in the first reason can be explained by shear locking of the bilinear elements as already noted. This error can be reduced by using more elements in the length direction or by introducing nonlinear element types.

3.4 Effects of Various Model Parameters. Parametric studies were performed to investigate the effects of various parameters on the first frequency and quality factor of resonance. Figure 4 displays the effect of beam aspect ratio. Clearly, increase in thickness causes an increase of stiffness, and thus raises the eigenfrequency. However, the quality factor has a minimum value at a certain aspect ratio and therefore special considerations should be taken for design of beams near this minimal Q. Figure 5 shows the quality factor/frequency as functions of temperature. It is evident that the thermoelastic damping does not significantly affect the resonant frequency, however, the quality factor decreases when the temperature increases. At high temperatures, the quality factor may drop significantly and therefore more energy will be dissipated compared to the low-temperature condition. This implies the importance of considering thermoelasticity in the design of MEMS devices working at high ambient temperatures.

Figure 6 shows the effect of thermal diffusivity. Again the changes in thermal properties do not have significant effects on the frequency of vibration; however, the quality factor can be affected remarkably, implying that it is essential to select materials of small thermal diffusivity in order to improve the quality factor. Figure 7 is the effect of Poisson's ratio. Clearly, the quality

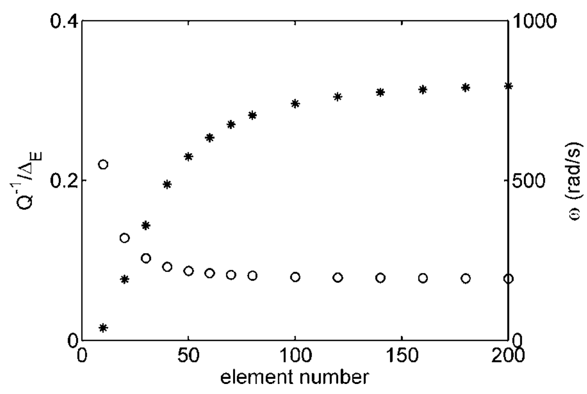

Fig. 2 Convergence test showing the quality factor (*) and frequency (○) as functions of element number

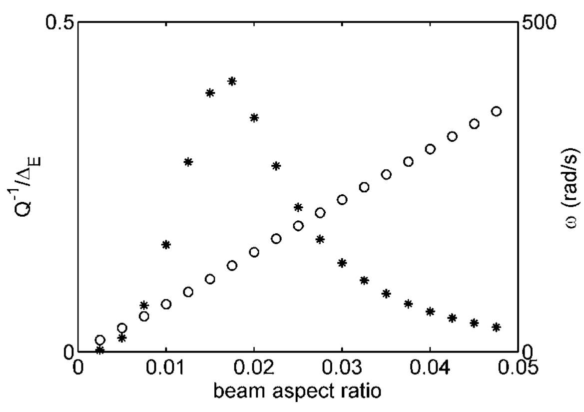

Fig. 4 Frequency (○) and quality factor (*) of vibration as functions of beam aspect ratio

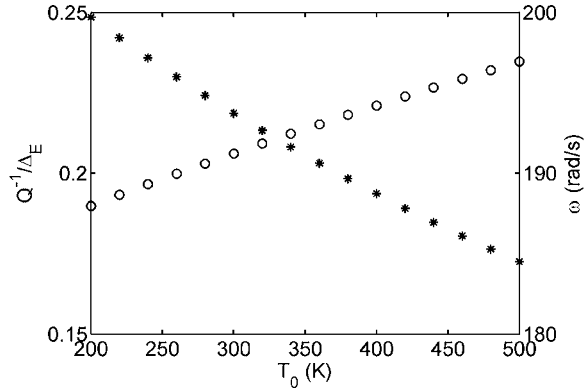

Fig. 5 Frequency (○) and quality factor (*) of vibration as functions of temperature

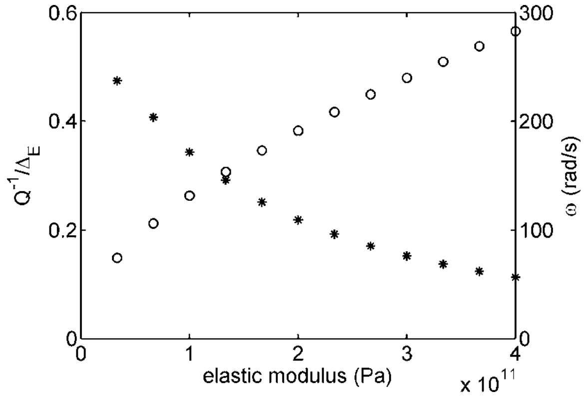

Fig. 8 Frequency (○) and quality factor (*) of vibration as functions of Young's modulus

factor increases with Poisson's ratio. In Fig. 8, the quality factor decreases when the material stiffness increases. All the above results are consistent with the existing analytical results found in the literature.

4 Conclusions

The thermoelastic damping problem involved in beam resonators has been solved using a combination of finite element method and eigenvalue formulation. By applying a small perturbation of the temperature, displacement, and velocity fields, the governing equations were reduced to an eigenvalue equation, in which the growth rate of the eigenvalue contains information leading to the frequency and damping factor of vibration. The finite element method was then applied to obtain an approximation solution to the eigenvalue equation. The numerical results obtained from the finite element method are in agreement with the existing analytical solutions from several different sources. Parametric studies have also shown the importance of a variety of geometric and material properties on the resonant frequency and the quality factor. These results are again consistent with the results reported by former researchers. Compared to analytical approaches of the problem, the finite element method is superior in that the solutions of the problem can be determined for complicated geometries and boundary conditions. For example, the same formulation can also be extended to the cases in which the geometry is three dimensional and the boundary conditions are nonuniform, where in most situations analytical solutions will be extremely difficult to obtain. Detailed discussion about this extension will possibly form a second paper on the work in the near future.

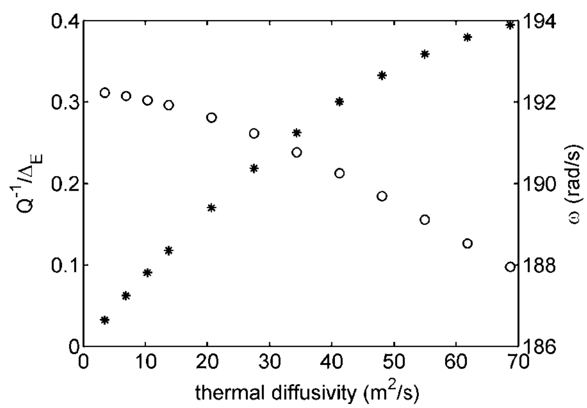

Fig. 6 Frequency (○) and quality factor (*) of vibration as functions of thermal diffusivity

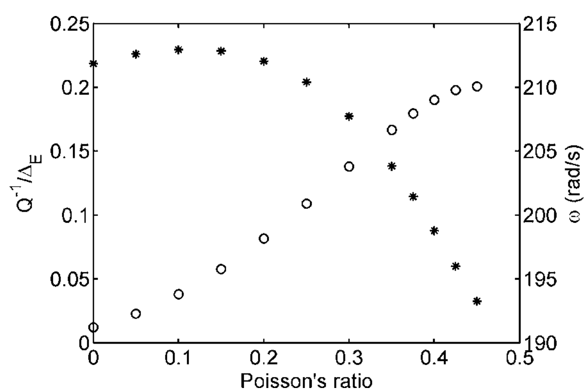

Fig. 7 Frequency (○) and quality factor (*) of vibration as functions of Poisson's ratio

Nomenclature

b	=	exponential decaying rate
C_p	=	specific heat capacity
e	=	dilatation strain
E	=	Young's modulus (or elastic modulus)
h	=	beam thickness
k	=	thermal conductivity
L	=	beam length
Q	=	quality factor
t	=	time
T	=	temperature
T_0	=	mean temperature
u	=	displacement
α	=	thermal expansion coefficient
ε	=	strain
σ	=	stress
ρ	=	density
ν	=	Poisson's ratio
Δ_E	=	relaxation strength of Young's modulus
τ	=	relaxation time
ω	=	natural frequency
ω_0	=	undamped natural frequency

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